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Bootstrap Determination of the Co-integration Rank in Heteroskedastic VAR Models*

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Abstract

In a recent paper Cavaliere *et al.* (2012) develop bootstrap implementations of the (pseudo-) likelihood ratio [PLR] co-integration rank test and associated sequential rank determination procedure of Johansen (1996). The bootstrap samples are constructed using the restricted parameter estimates of the underlying VAR model which obtain under the reduced rank null hypothesis. They propose methods based on an i.i.d. bootstrap re-sampling scheme and establish the validity of their proposed bootstrap procedures in the context of a co-integrated VAR model with i.i.d. innovations. In this paper we investigate the properties of their bootstrap procedures, together with analogous procedures based on a wild bootstrap re-sampling scheme, when time-varying behaviour is present in either the conditional or unconditional variance of the innovations. We show that the bootstrap PLR tests are asymptotically correctly sized and, moreover, that the probability that the associated bootstrap sequential procedures select a rank smaller than the true rank converges to zero. This result is shown to hold for both the i.i.d. and wild bootstrap variants under conditional heteroskedasticity but only for the latter under unconditional heteroskedasticity. Monte Carlo evidence is reported which suggests that the bootstrap approach of Cavaliere *et al.* (2012) significantly improves upon the finite sample performance of corresponding procedures based on either the asymptotic PLR test or an alternative bootstrap method (where the short run dynamics in the VAR model are estimated unrestrictedly) for a variety of conditionally and unconditionally heteroskedastic innovation processes.

Keywords: Bootstrap; Co-integration; Trace statistic; Rank determination; heteroskedasticity.

J.E.L. Classifications: C30, C32.

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1 Introduction

Sequential likelihood-based procedures for the determination of the co-integration rank in VAR systems of variables integrated of order 1 [$I(1)$], see Johansen (1996), are widely used in empirical research. However, it is now well understood that the finite sample properties of these procedures, when based on asymptotic inference, can be quite poor; see, in particular, Johansen (2002) and the references therein. It is also well-known that the bootstrap, when correctly implemented, can be an important device to compute critical values of asymptotic tests in samples of finite size thereby delivering tests with empirical rejection frequencies closer to the nominal level. As a consequence, it is not surprising that there has been an increasing interest in using bootstrap methods in determining the co-integration rank in vector autoregressive models. For co-integrated VAR models with independent and identically distributed (i.i.d.) innovations, see, most notably, Swensen (2006) and Cavaliere, Rahbek and Taylor (2012); for VAR models with potentially heteroskedastic innovations, see Cavaliere, Rahbek and Taylor (2010*a*, 2010*b*).

A key feature of the bootstrap algorithms proposed in Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*) is that they combine restricted (where the null co-integrating rank is imposed) estimates of the long-run parameters of the model with unrestricted parameter estimates of the short run parameters in the bootstrap recursion used to generate the bootstrap sample data. As is recognised in Swensen (2009), where the null hypothesis imposes a co-integration rank r which is smaller than the true rank, r_0 say, the potential arises for the resulting bootstrap samples to be non-$I(1)$ (they can, for example, be explosive or admit too many roots on the unit circle), thereby invalidating the use of the bootstrap, even asymptotically. Swensen (2009) shows that for this not to happen a number of auxiliary conditions must be imposed on the (unknown) parameters of the data generating process (DGP).

In a recent paper, for the case of co-integrated VAR models with i.i.d. innovations, Cavaliere *et al.* (2012) [CRT hereafter] show that this problem can be solved by considering an alternative bootstrap scheme where the bootstrap recursion uses parameter estimates of the short run and long run parameters both of which are obtained under the null co-integrating rank. CRT demonstrate that even when $r < r_0$ these estimates converge to pseudo-true values which ensure that the resulting bootstrap data are (at least in large samples) $I(1)$ with co-integrating rank r. As a consequence they show that the resulting bootstrap tests are asymptotically valid, attaining the same first-order limit null distribution as the original pseudo likelihood ratio [PLR] statistic both when $r = r_0$ and, crucially, when $r < r_0$, without the need for any auxiliary conditions to hold on the underlying DGP. Given that the PLR statistic diverges when $r < r_0$ they then show that this result ensures that the associated bootstrap analogue of Johansen's sequential procedure is consistent in the usual sense that the probability of choosing a rank smaller than the true rank will converge to zero. Like Swensen (2006), the procedures proposed in CRT are based on an i.i.d. re-sampling scheme.

In this paper we analyse the properties of the bootstrap PLR tests and associated sequential procedures proposed in CRT in cases where the innovations may display time-varying behaviour in either their conditional or unconditional variances, of the form considered in Cavaliere *et al.* (2010a, 2010b). The former, embodied in a martingale difference assumption, permits, for example, certain types of GARCH models for the volatility process, while the latter allows, for example, single and multiple abrupt (co-)variance breaks, smooth transition (co-)variance breaks, and trending (co-)variances. In our analysis we will consider procedures based on an i.i.d. bootstrap re-sampling scheme, as outlined in CRT, together with analogous procedures based on a wild bootstrap re-sampling scheme. We show that the wild bootstrap analogues of the algorithms proposed in CRT are asymptotically valid in both cases, again attaining the same first-order limit null distribution as the PLR statistic when $r \leq r_0$, and again without the need for any auxiliary conditions to hold on the underlying DGP. The same result is shown to hold for the i.i.d. bootstrap implementation of the algorithms for the conditionally heteroskedastic case considered. In contrast, in the non-constant volatility case this result is only attainable using the wild bootstrap versions of CRT's algorithms. These are particularly useful results since the Bartlett-corrected rank tests of Johansen (2002), which constitute an alternative approach to the bootstrap to improve the finite sample properties of the tests, are not appropriate when the errors are heteroskedastic.

The paper is organised as follows. Section 2 outlines our heteroskedastic co-integrated VAR model. Section 3 outlines the pseudo-LR co-integration rank tests and associated sequential procedures of Johansen (1996), outlining the large sample properties of these under heteroskedastic innovations. The bootstrap algorithms proposed by CRT are outlined in section 4, and here it is briefly shown how these differ from the corresponding bootstrap algorithms from Swensen (2006). The large sample properties of the bootstrap procedures under heteroskedastic innovations are established in section 5. The results of a Monte Carlo study are given in section 6. Section 7 concludes. Mathematical proofs are contained in the Appendix.

In the following $\xrightarrow{w}$ denotes weak convergence, $\xrightarrow{p}$ convergence in probability, and $\xrightarrow{w}_p$ weak convergence in probability (Giné and Zinn, 1990; Hansen, 1996), in each case as $T \to \infty$; $\mathbb{I}(\cdot)$ denotes the indicator function; $x := y$ indicates that x is defined by y; $\lfloor \cdot \rfloor$ denotes the integer part of its argument; $\mathcal{C}_{\mathbb{R}^{m\times n}}[0,1]$ denotes the space of $m \times n$ matrices of continuous functions on $[0,1]$; $\mathcal{D}_{\mathbb{R}^{m\times n}}[0,1]$ denotes the space of $m \times n$ matrices of càdlàg functions on $[0,1]$; I_k denotes the $k \times k$ identity matrix and $0_{j\times k}$ the $j \times k$ matrix of zeroes; the space spanned by the columns of any $m \times n$ matrix a is denoted as $\mathsf{col}(a)$; if a is of full column rank $n < m$, then $\bar{a} := a\,(a'a)^{-1}$ and $a_\perp$ is an $m \times (m-n)$ full column rank matrix satisfying $a'_\perp a = 0$; for any square matrix, a, $|a|$ is used to denote its determinant, $\|a\|$ the norm $\|a\|^2 := \mathsf{tr}\{a'a\}$ and $\rho(a)$ its spectral radius (that is, the maximal modulus of the eigenvalues of a); for any vector, x, $\|x\|$ denotes the usual Euclidean norm, $\|x\| := (x'x)^{1/2}$. Finally, P^* denotes the bootstrap probability measure, i.e. conditional on the original sample; similarly, E^* denotes expectation under P^*.

2 The Heteroskedastic Co-integrated VAR Model

Following Johansen (1996), we consider the case where the p-dimensional observations $\{X_t\}$ satisfy the kth order reduced rank vector autoregressive (VAR) model

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \alpha\rho' D_t + \phi d_t + \varepsilon_t, \quad t = 1, ..., T \tag{2.1}$$

where $X_t := (X_{1t}, ..., X_{pt})'$, $\varepsilon_t := (\varepsilon_{1t}, ..., \varepsilon_{pt})'$, and where the initial values, $X_{1-k}, ..., X_0$, are taken to be fixed in the statistical analysis. The deterministic variables are assumed to satisfy one of the following cases (see, e.g., Johansen, 1996): (i) $d_t = 0$, $D_t = 0$ (no deterministic component); (ii) $D_t = 1$, $d_t = 0$ (restricted constant), or (iii) $D_t = t$, $d_t = 1$ (restricted linear trend). The innovation process $\{\varepsilon_t\}$ is taken to satisfy one of the following three assumptions:

Assumption $\mathcal{V}$ *The innovations $\{\varepsilon_t\}$ are independent and identically distributed with mean zero and full-rank variance matrix Σ, and where $\mathrm{E}\,\|\varepsilon_t\|^4 \leq K < \infty$.*

Assumption $\mathcal{V}$' *The innovations $\{\varepsilon_t\}$ form a martingale difference sequence with respect to the filtration $\mathcal{F}_t$, where $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$ for $t = ..., -1, 0, 1, 2, ...$, satisfying: (i) the global homoskedasticity condition:*

$$\frac{1}{T}\sum_{t=1}^{T} \mathrm{E}\left(\varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1}\right) \xrightarrow{p} \Sigma > 0, \tag{2.2}$$

and (ii) $\mathrm{E}\,\|\varepsilon_t\|^4 \leq K < \infty$.

Assumption $\mathcal{V}$" *The innovations $\{\varepsilon_t\}$ are such that $\varepsilon_t = \sigma_t z_t$, where z_t is p-variate i.i.d., $z_t \sim (0, I_p)$ with $\mathrm{E}\,\|z_t\|^4 \leq K < \infty$, and where the matrix σ_t is non-stochastic and satisfies $\sigma_t := \sigma(t/T)$ for all $t = 1, ..., T$, where $\sigma(\cdot) \in \mathcal{D}_{\mathbb{R}^{p\times p}}[0, 1]$. Moreover it is assumed that $\Sigma(u) := \sigma(u)\,\sigma(u)'$ is positive definite for all $u \in [0, 1]$.*

Remark 1 Assumption $\mathcal{V}$ is that considered by Johansen (1996) and Swensen (1996). Assumption $\mathcal{V}$' is taken from Cavaliere *et al.* (2010*a*) and allows for, among other things, models with deterministic periodic heteroskedasticity and for multivariate versions of the stable GARCH, EGARCH, AGARCH, GJR-GARCH, and autoregressive stochastic volatility models of the type considered in Gonçalves and Kilian (2004,p.99); see also Section 6. Notice that condition (i) of Assumption $\mathcal{V}$' imposes neither strict nor second-order stationarity on ε_t, but rather imposes a so-called *global stationarity* or *global homoskedasticity* condition; see e.g. Davidson (1994,pp.454-455). Assumption $\mathcal{V}$" implies that the elements of the innovation covariance matrix $\Sigma_t := \sigma_t \sigma_t'$ are only required to be bounded and to display a countable number of jumps, therefore allowing for an extremely wide class of potential models for the behaviour of the covariance matrix of ε_t. Models of single or multiple variance or covariance shifts, satisfy Assumption $\mathcal{V}$" with $\Sigma(\cdot)$ piecewise constant. For instance, denoting the (i, j)th element of $\Sigma(u)$ by $\Sigma_{ij}(u)$, the case of a single break at time $\lfloor \tau T \rfloor$ in the covariance $E(\varepsilon_{it}\varepsilon_{jt})$ obtains for $\Sigma_{ij}(u) = \Sigma_{ij}^0 + (\Sigma_{ij}^1 - \Sigma_{ij}^0)\mathbb{I}(u \geq \tau)$. Piecewise affine functions are also permitted, thereby allowing for variances which follow a (possibly) broken trend, as are smooth transition variance shifts.

The requirement within Assumption $\mathcal{V}$"that $\sigma(\cdot)$ is non-stochastic is made in order to simplify the analysis, but can be generalised to allow for cases where $\sigma(\cdot)$ is stochastic and independent of z_t; see Remark 2.2 of Cavaliere *et al.* (2010*b*) for further details.

In what follows we will often refer to the case where the parameters of (2.1) satisfy the '$I(1,r)$ conditions'. These are formally defined below.

Definition 1 *If: (a) the characteristic polynomial associated with* (2.1) *has* $p-r$ *roots equal to 1 and all other roots outside the unit circle, and (b)* α *and* β *have full column rank* r*, then the parameters in* (2.1) *will be said to satisfy the '*$I(1,r)$ *conditions'.*

Under the conditions given in Definition 1, and coupled with either Assumption $\mathcal{V}$, $\mathcal{V}$' or $\mathcal{V}$", X_t is $I(1)$ with co-integration rank r. Here we are using the definition of $I(1)$ adopted by Cavaliere *et al.* (2010*b*) which is defined such that the common trend component of the data admits a functional central limit theorem. Under conditions (a) and (b) of Definition 1 and if Assumption $\mathcal{V}$ holds, then the co-integrating relations $\beta' X_t - E\left(\beta' X_t\right)$ will then be stationary, while under Assumption $\mathcal{V}$' they will be globally stationary. Under Assumption $\mathcal{V}$", however, stationarity does not hold in general on the co-integrating relations, due to the time-variation present in σ_t; nonetheless, $\beta' X_t - E\left(\beta' X_t\right)$ is stable, in the sense that it is free of stochastic trends.

3 Pseudo Likelihood Ratio Tests

The well-known PLR test[1] of Johansen (1996) for the hypothesis of co-integration rank (less than or equal to) r in (2.1), denoted $H(r)$, against $H(p)$, rejects for large values of the trace statistic, $Q_{r,T} := -T\sum_{i=r+1}^{p}\log(1-\hat{\lambda}_i)$, where $\hat{\lambda}_1 > \ldots > \hat{\lambda}_p$ are the largest p solutions to the eigenvalue problem,

$$\left|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}\right| = 0, \tag{3.1}$$

where $S_{ij} := T^{-1}\sum_{t=1}^{T} R_{it}R_{jt}'$, $i,j = 0,1$, with R_{0t} and R_{1t} respectively denoting ΔX_t and $\left(X_{t-1}', D_t\right)'$, corrected (by OLS) for $\Delta X_{t-1}, ..., \Delta X_{t-k+1}$ and d_t. The sequential testing procedure based on $Q_{r,T}$ involves, starting with $r=0$, testing in turn $H(r)$ against $H(p)$ for, $r = 0, ..., p-1$, until, for a given value of r, the asymptotic p-value associated with $Q_{r,T}$, exceeds a chosen (marginal) significance level.

Suppose that X_t in (2.1) satisfies Definition 1 for $r = r_0$; that is, the true co-integrating rank is r_0 and (2.1) satisfies the $I(1, r_0)$ conditions. Then, under either Assumption $\mathcal{V}$ (see Johansen, 1996) or Assumption $\mathcal{V}$' (see, Cavaliere *et al.*, 2010*a*) it holds that

$$Q_{r_0,T} \xrightarrow{w} \mathsf{tr}(\mathcal{Q}_{r_0,\infty}) \tag{3.2}$$

[1] By which we mean the test based on the likelihood which obtains under the assumption that ε_t in (2.1) are Gaussian i.i.d. disturbances. The associated estimators from (2.1) under this assumption will, correspondingly, be referred to as pseudo maximum likelihood estimators.

where, for a generic argument r,

$$\mathcal{Q}_{r,\infty} := \int_0^1 dB_{p-r}(u)F_{p-r}(u)' \left(\int_0^1 F_{p-r}(u)F_{p-r}(u)'du\right)^{-1} \int_0^1 F_{p-r}(u)dB_{p-r}(u)' \tag{3.3}$$

where $B_{p-r}(\cdot)$ is a $(p-r)$-variate standard Brownian motion, and where either: (i) in the no deterministics case, $F_{p-r} := B_{p-r}$; (ii) in the restricted constant case, $F_{p-r} := (B'_{p-r}, 1)'$, or, (iii) in the restricted linear trend case, $F_{p-r} := (B'_{p-r}, u|1)'$, where $a|b := a(\cdot) - \int a(s)b(s)'ds(\int b(s)b(s)'ds)^{-1}b(\cdot)$ denotes the projection residuals of a onto b. Critical values from these (pivotal) limiting null distributions are provided in Johansen (1996). Under Assumption $\mathcal{V}$", however, Cavaliere *et al.* (2010*b*) establish that

$$Q_{r_0,T} \xrightarrow{w} \mathsf{tr}(\mathcal{Q}^H_{r_0,\infty}) \tag{3.4}$$

where, again for a generic argument r, $\mathcal{Q}^H_{r,\infty}$ is defined by the right hand side of (3.3) but replacing the standard Brownian motion, $B_{p-r}(u)$, throughout by the the $(p-r)$-variate stochastic volatility process

$$\tilde{M}_{p-r}(u) := \left(\alpha'_\perp \Sigma \alpha_\perp\right)^{-1/2} \alpha'_\perp \int_0^u \sigma(s)dB_p(s)$$

where $\Sigma := \int_0^1 \Sigma(s)\, ds$ is the (asymptotic) average innovation variance. This limiting null distribution is in general non-pivotal, its form depending on the spot volatility process, $\sigma(\cdot)$. Consequently, inference using the standard trace statistics will not in general be pivotal under Assumption $\mathcal{V}$" if p-values are retrieved on the basis of the tabulated distributions which apply under Assumptions $\mathcal{V}$ and $\mathcal{V}$'.

Under the $I(1, r_0)$ conditions, and regardless of whether Assumption $\mathcal{V}$, $\mathcal{V}$' or $\mathcal{V}$" holds, the r_0 largest eigenvalues solving (3.1), $\hat{\lambda}_1, \ldots, \hat{\lambda}_{r_0}$, converge in probability to positive numbers, while $T\hat{\lambda}_{r_0+1}, \ldots, T\hat{\lambda}_p$ are of $O_p(1)$. Consequently, under any of Assumptions $\mathcal{V}$, $\mathcal{V}$' or $\mathcal{V}$" the standard asymptotic test based on $Q_{r,T}$ will be consistent at rate $O_p(T)$ if r_0 is such that $r_0 > r$. This implies, therefore, that under either Assumption $\mathcal{V}$ or $\mathcal{V}$' the sequential approach to determining the co-integration rank outlined above will be consistent in the usual sense that it will lead to the selection of the correct co-integrating rank with probability $(1-\xi)$ in large samples if a marginal significance level of ξ is chosen. However, under Assumption $\mathcal{V}$" this will not in general be true, unless critical values from the limiting distribution on the right hand side of (3.2) are used: the standard sequential approach to determining the co-integration rank will therefore not in general lead to the selection of the correct co-integrating rank with probability $(1-\xi)$ even in large samples.

To conclude this section we detail the large sample properties of the pseudo maximum likelihood estimates [PMLE] of the parameters of (2.1) that obtain under $H(r)$ under each of Assumptions $\mathcal{V}$, $\mathcal{V}$' and $\mathcal{V}$". To that end, let $\hat{v} := (\hat{v}_1, \hat{v}_2, ..., \hat{v}_p)$ denote the eigenvectors from (3.1), viz,

$$\hat{v}'S_{11}\hat{v} = I_p, \qquad \hat{v}'S_{10}S_{00}^{-1}S_{01}\hat{v} = \hat{\Lambda}_p := \mathrm{diag}(\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_p)\ . \tag{3.5}$$

The (uniquely defined) Gaussian PMLE of β, $\hat{\beta}^{(r)}$, may then be written as $\hat{\beta}^{(r)} := \hat{v}K_p^{(r)}$, where $K_p^{(r)} := (I_r, 0_{r\times(p-r)})'$, is a selection matrix indexed by r and p. When deterministic terms are included, $\hat{\beta}^{\#(r)} := (\hat{\beta}^{(r)\prime}, \hat{\rho}^{(r)\prime})' = \hat{v}K_{p+1}^{(r)}$. The remaining estimators $\hat{\alpha}^{(r)}$,$\widehat{\Gamma}_1^{(r)}, \ldots, \widehat{\Gamma}_{k-1}^{(r)}$ and $\hat{\phi}^{(r)}$ are then obtained by OLS regression, as in Johansen (1996). We denote the PMLE for (2.1) under $H(r)$ by $\hat{\theta}^{(r)} := \{\hat{\alpha}^{(r)}, \hat{\beta}^{(r)}, \widehat{\Gamma}_1^{(r)}, ..., \widehat{\Gamma}_{k-1}^{(r)}, \hat{\rho}^{(r)}, \hat{\phi}^{(r)}, \hat{\Sigma}^{(r)}\}$.

Under the $I(1, r_0)$ conditions, Johansen (1996) establishes that under Assumption $\mathcal{V}$, $\hat{\theta}^{(r)} \xrightarrow{p} \theta$, where $\theta := \{\alpha, \beta, \Gamma_1, ..., \Gamma_{k-1}, \rho, \phi, \Sigma\}$. Cavaliere *et al.* (2010a) show that this result also holds under Assumption $\mathcal{V}$', provided Σ is defined as in (2.2). Under Assumption $\mathcal{V}$", Cavaliere *et al.* (2010b) prove that, again under the $I(1, r_0)$ conditions, $\hat{\theta}^{(r)} \xrightarrow{p} \theta_*$, where $\theta_* := \{\alpha, \beta, \Gamma_1, ..., \Gamma_{k-1}, \rho, \phi, \Sigma\}$ with Σ now equal to $\int_0^1 \Sigma(s)\,ds$. Under Assumption $\mathcal{V}$, CRT demonstrate the important additional result, which is pivotal to showing the validity of their proposed bootstrap procedure, that when $H(r)$ imposes a co-integration rank which is smaller than the true rank, r_0 say, such that $\hat{\theta}^{(r)} \xrightarrow{p} \theta_0^{(r)}$, where $\theta_0^{(r)} := \{\alpha_0^{(r)}, \beta_0^{(r)}, \Gamma_{1,0}^{(r)}, ..., \Gamma_{k-1,0}^{(r)}, \rho_0^{(r)}, \phi_0^{(r)}, \Sigma_0^{(r)}\}$ is a vector of pseudo-true parameters which have the key property that they satisfy the $I(1, r)$ conditions. In section 5 we will show that this result also holds under either Assumption $\mathcal{V}$' or $\mathcal{V}$", allowing us to establish the asymptotic validity of the bootstrap PLR tests and associated sequential procedure proposed in CRT, adapted to use a wild bootstrap re-sampling scheme where appropriate, when the innovations are heteroskedastic.

4 Bootstrap Algorithms

In Algorithm 1 we detail the bootstrap implementation of the PLR test for $H(r)$ against $H(p)$. Where the i.i.d. re-sampling scheme, (a), is adopted in step (iii), this algorithm coincides with Algorithm 1 of CRT.

Algorithm 1:

(i) Estimate model (2.1) under $H(r)$ using Gaussian PMLE yielding the estimates $\hat{\beta}^{(r)}$, $\hat{\alpha}^{(r)}, \hat{\rho}^{(r)}$, $\widehat{\Gamma}_1^{(r)}, ..., \widehat{\Gamma}_{k-1}^{(r)}$ and $\phi^{(r)}$, together with the corresponding residuals, $\hat{\varepsilon}_{r,t}$.

(ii) Check that the equation $|\hat{A}^{(r)}(z)| = 0$, with $\hat{A}^{(r)}(z) := (1-z) I_p - \hat{\alpha}^{(r)} \hat{\beta}^{(r)\prime} z - \sum_{i=1}^{k-1} \hat{\Gamma}_i^{(r)} (1-z) z^i$, has $p - r$ roots equal to 1 and all other roots outside the unit circle. If so, proceed to step (iii).

(iii) Construct the bootstrap sample recursively from

$$\Delta X_{r,t}^* = \hat{\alpha}^{(r)} \hat{\beta}^{(r)\prime} X_{r,t-1}^* + \sum_{i=1}^{k-1} \widehat{\Gamma}_i^{(r)} \Delta X_{r,t-i}^* + \hat{\alpha}^{(r)} \hat{\rho}^{(r)\prime} D_t + \hat{\phi}^{(r)} d_t + \varepsilon_{r,t}^*, \ t = 1, ..., T \qquad (4.1)$$

initialised at $X_{r,j}^* = X_j$, $j = 1-k, ..., 0$, and with the T bootstrap errors $\varepsilon_{r,t}^*$ generated using the re-centred residuals, $\hat{\varepsilon}_{r,t}^c := \hat{\varepsilon}_{r,t} - T^{-1} \sum_{i=1}^{T} \hat{\varepsilon}_{r,i}$, for either:

(a) the i.i.d. bootstrap, such that $\varepsilon_{r,t}^* := \hat{\varepsilon}_{r,\mathcal{U}_t}^c$, where $\mathcal{U}_t$, $t = 1, ..., T$ is an i.i.d. sequence of discrete uniform distributions on $\{1, 2, ..., T\}$, or

(b) the wild bootstrap, where for each $t = 1, ..., T$, $\varepsilon_{r,t}^* := \hat{\varepsilon}_{r,t}^c w_t$, where w_t, $t = 1, ..., T$, is an i.i.d. N(0,1) sequence.

(iv) Using the bootstrap sample, $\{X^*_{r,t}\}$, and denoting by $\hat{\lambda}^*_1 > \ldots > \hat{\lambda}^*_p$ the ordered solutions to the bootstrap analogue of the eigenvalue problem in (3.1), compute the bootstrap LR statistic $Q^*_{r,T} := -T\sum_{i=r+1}^{p}\log(1-\hat{\lambda}^*_i)$. Define the corresponding p-value as $p^*_{r,T} := 1 - G^*_{r,T}(Q_{r,T})$, $G^*_{r,T}(\cdot)$ denoting the conditional (on the original data) cdf of $Q^*_{r,T}$.

(v) The bootstrap test of $H(r)$ against $H(p)$ at level η rejects $H(r)$ if $p^*_{r,T} \le \eta$.

Remark 2 The recursive scheme in (4.1) differs from the corresponding bootstrap recursion in Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*) which takes the form

$$\Delta X^*_{r,t} = \hat{\alpha}^{(r)}\hat{\beta}^{(r)\prime}X^*_{r,t-1} + \sum_{i=1}^{k-1}\widehat{\Gamma}^{(p)}_i\Delta X^*_{r,t-i} + \hat{\alpha}^{(r)}\hat{\rho}^{(r)\prime}D_t + \hat{\phi}^{(p)}d_t + \varepsilon^*_{p,t},\ t = 1, ..., T \tag{4.2}$$

where $\widehat{\Gamma}^{(p)}_1, \ldots, \widehat{\Gamma}^{(p)}_{k-1}$ and $\hat{\phi}^{(p)}$ are now the estimates of the short run matrices $\Gamma_1, \ldots, \Gamma_{k-1}$ and ϕ, respectively, from estimating (2.1) unrestrictedly, i.e. under $H(p)$. This difference is crucial since showing that the bootstrap test of $H(r)$ is consistent when $r < r_0$, requires that the bootstrap data still satisfy the $I(1, r)$ conditions in large samples. As acknowledged in Swensen (2009), this is not guaranteed, even asymptotically, when using the recursion in (4.2), unless a number of auxiliary restrictions, labelled Assumption 2 in Swensen (2009), hold on the parameters of (2.1); see also Remark 6 of CRT. CRT show that these restrictions are not needed if the bootstrap recursion in (4.1) is used since it always delivers an $I(1)$ system with $r \le r_0$ co-integrating vectors in the limit, regardless of the true co-integration rank, r_0.

Remark 3 Although, as CRT show, Algorithm 1 without the inclusion of step (ii) ensures that the bootstrap data satisfy the $I(1, r)$ conditions in the limit, this could fail in small samples. Consequently, the role of step (ii) is to check that the bootstrap samples will indeed be $I(1)$ with co-integration rank r. Unreported simulations for the case where we continue to step (iii) of Algorithm 1 regardless of whether the root check condition in step (ii) is failed or not suggest, reassuringly, that this leads to no deterioration in the finite sample performance of the resulting bootstrap tests relative to the results reported here. Analogous conditions to those in step (ii) are also checked in step (iii) of Algorithm 1 in Swensen (2006) for the recursion in (4.2). Notice that step (ii) will be failed with probability one in Algorithm 1 of Swensen (2006) unless Assumption 2 of Swensen (2009) is satisfied.

Remark 4 In practice, the cdf $G^*_{r,T}(\cdot)$ required in Step (iv) of Algorithm 1 will be unknown, but can be approximated in the usual way through numerical simulation; see, *inter alia*, Hansen (1996), Davidson and MacKinnon (2000) and Andrews and Buchinsky (2000). This is achieved by generating B (conditionally) independent bootstrap statistics, $Q^*_{r,T:b}$, $b = 1, ..., B$, computed as in Algorithm 1 above. The simulated bootstrap p-value is then computed as $\tilde{p}^*_{r,T} := B^{-1}\sum_{b=1}^{B}\mathbb{I}(Q^*_{r,T:b} > Q_{r,T})$, and is such that $\tilde{p}^*_{r,T} \overset{a.s.}{\to} p^*_{r,T}$ as $B \to \infty$.

We conclude this section by outlining in Algorithm 2 the bootstrap sequential algorithm for determining the co-integrating rank. Again for re-sampling scheme (a) in step (iii) of Algorithm 1, this replicates Algorithm 2 of CRT.

Algorithm 2: Starting from $r = 0$ perform the following steps:

(i)–(iv) Same as in Algorithm 1.

(v) If $p^*_{r,T}$ exceeds the significance level, η, set $\hat{r} = r$, otherwise repeat steps (i)–(iv) testing the null of rank $(r+1)$ against rank p if $r+1 < p$, or set $\hat{r} = p$ if $r+1 = p$.

5 Asymptotic Analysis

CRT establish the large sample behaviour of the PLR tests and associated sequential procedure from Algorithms 1 and 2 respectively, for any $I(1)$ DGP satisfying the conditions stated in Swensen (2006). These conditions are comprised of those made for the standard asymptotic test in Johansen (1996), Assumption 1 below and Assumption $\mathcal{V}$, coupled with the additional assumption from Swensen (2006, Lemma 3), stated as Assumption 2 below, that eigenvalues from (3.1) are distinct in the limit. Crucially, CRT show that the asymptotic validity of their bootstrap procedures do not require any further conditions, such as Assumption 2 of Swensen (2009), to hold.

Assumption 1 *The parameters in (1) satisfy the I(1,r_0) conditions.*

Assumption 2 *The limiting non-zero roots of* (3.1) *are distinct.*

Precisely, CRT demonstrate that, under Assumption $\mathcal{V}$ and Assumptions 1 and 2, for any $r \le r_0$, $Q^*_{r,T} \xrightarrow{w}_p \mathsf{tr}(\mathcal{Q}_{r,\infty})$, where $\mathcal{Q}_{r,\infty}$ is as defined in (3.3), for the case where the i.i.d. bootstrap re-sampling design is used in step (iii) of Algorithm 1. It is straightforward to show that the same result holds where the wild bootstrap is used in step (iii) of Algorithm 1; indeed this results follows as a special case of the results given below noting that Assumption $\mathcal{V}$ is a special case of both Assumption $\mathcal{V}$' and Assumption $\mathcal{V}$". An immediate consequence of this result and the results previously detailed for the asymptotic PLR tests, is that the bootstrap test based on $Q^*_{r,T}$ will be asymptotically correctly sized under the null hypothesis ($r = r_0$), and will be consistent for all $r < r_0$; that is, $p^*_{r_0,T} \xrightarrow{w} U[0,1]$ and[2] $p^*_{r,T} := 1 - G^*_{r,T}(Q_{r,T}) \xrightarrow{p} 0$, for all $r < r_0$. CRT show that these results obtain by virtue of the fact that under Assumption $\mathcal{V}$ the PMLE, $\hat{\theta}^{(r)}$, used to generate the bootstrap samples in Algorithm 1 asymptotically satisfies the $I(1,r)$ conditions, even when an incorrect rank $r < r_0$ is imposed.

Our aim in this section is to prove that the bootstrap PLR tests from Algorithm 1 are also asymptotically valid under either Assumption $\mathcal{V}$' or Assumption $\mathcal{V}$", i.e. that they are asymptotically correctly sized under the null and consistent under the alternative in the presence of heteroskedasticity in the innovations.[3] In the light of the results in CRT it is clear that in order to do so we must first

[2] Notice that if the p-value of a test converges in large samples to a uniform distribution on $[0,1]$ under the null hypothesis, then for any chosen significance level η, as the sample size diverges the probability of rejecting the null hypothesis converges to η; i.e., the test has asymptotic size η, as required.

[3] It is worth noting that the large sample results that we establish for the wild bootstrap version of $Q^*_{r,T}$ in this section are obtained under weaker conditions than were required in Cavaliere et al (2010b) who, in deriving the large sample properties of their proposed wild bootstrap test, additionally required the innovations, z_t, in Assumption $\mathcal{V}$" to be symmetrically distributed.

establish that $\hat{\theta}^{(r)}$ also satisfies the $I(1,r)$ conditions in the limit, even when an incorrect rank $r < r_0$ is imposed, under both Assumption $\mathcal{V}$' and $\mathcal{V}$". This is done in Lemma 1.

Lemma 1 *Let $\{X_t\}$ be generated as in (2.1) under Assumptions 1, 2 and either $\mathcal{V}$' or $\mathcal{V}$". Then: (i) for any $r \leq r_0$ and as $T \to \infty$, $\hat{\theta}^{(r)} \xrightarrow{p} \theta_0^{(r)}$, with the vector of pseudo-true parameters, $\theta_0^{(r)} := \{\alpha_0^{(r)}, \beta_0^{(r)}, \Gamma_{1,0}^{(r)}, ..., \Gamma_{k-1,0}^{(r)}, \rho_0^{(r)}, \phi_0^{(r)}, \Sigma_0^{(r)}\}$, defined in the Appendix; and (ii) the pseudo-true parameters $\theta_0^{(r)}$ satisfy the $I(1,r)$ conditions.*

An important consequence of Lemma 1 for the bootstrap recursion in (4.1) is stated in the following proposition, which establishes that for any $r \leq r_0$ the bootstrap sample generated by (4.1) is $I(1)$ with co-integration rank r in large samples.

Proposition 1 *Let $\{X_t\}$ be generated as in (2.1) under Assumptions 1 and 2, and let the bootstrap sample be generated as in Algorithm 1, for any $r \leq r_0$. Then the following results hold:*
(i) Under either Assumption $\mathcal{V}$' or $\mathcal{V}$", and for either the i.i.d. or wild bootstrap re-sampling design in step (iii) of Algorithm 1,

$$X_{r,t}^* = \hat{C}^{(r)} \sum_{i=1}^{t} \varepsilon_{r,i}^* + \hat{\tau}_{r,t} + S_{r,t} T^{1/2} \ , \tag{5.1}$$

where $\hat{C}^{(r)} := \hat{\beta}_{\perp}^{(r)} \left(\hat{\alpha}_{\perp}^{(r)\prime} \hat{\Gamma}^{(r)} \hat{\beta}_{\perp}^{(r)} \right)^{-1} \hat{\alpha}_{\perp}^{(r)\prime}$ with $\hat{\Gamma}^{(r)} := \sum_{i=1}^{k-1} \hat{\Gamma}_i^{(r)} - I$, and where $S_{r,t}$ is such that $P^* \left(\max_{t=1,...,T} \|S_{r,t}\| > \epsilon \right) \xrightarrow{p} 0$ for all $\epsilon > 0$. If there are either no deterministics or a restricted constant in (2.1) (i.e. cases (i) and (ii)), then $\hat{\tau}_{r,t} = 0$, while in the restricted linear trend case (case (iii)), $T^{-1} \hat{\tau}_{r,\lfloor Tu \rfloor} \xrightarrow{w} \tau_0^{(r)} u$, where $\tau_0^{(r)} := C_0^{(r)} \phi_0^{(r)} + (C_0^{(r)} \Gamma_0^{(r)} - I_p) \bar{\beta}_0^{(r)} \rho_0^{(r)\prime}$, where $C_0^{(r)} := \beta_{0\perp}^{(r)} (\alpha_{0\perp}^{(r)\prime} \Gamma_0^{(r)} \beta_{0\perp}^{(r)})^{-1} \alpha_{0\perp}^{(r)\prime}$ is of rank $(p-r) \geq (p-r_0)$, and where $\Gamma_0^{(r)} := \sum_{i=1}^{k-1} \Gamma_{0,i}^{(r)} - I_p$.

(ii) Under Assumption $\mathcal{V}$', for either the i.i.d. or wild bootstrap re-sampling design in step (iii) of Algorithm 1,

$$T^{-1/2} \hat{C}^{(r)} \sum_{i=1}^{\lfloor Tu \rfloor} \varepsilon_{r,i}^* \xrightarrow{w}_p C_0^{(r)} W_p(u) \, , \quad u \in [0,1] \tag{5.2}$$

where W_p is a p-dimensional Brownian motion with covariance matrix $\Omega_0^{(r)}$, and where $C_0^{(r)}$ is as defined in part (i).

(iii) Under Assumption $\mathcal{V}$", for the wild bootstrap re-sampling design in step (iii)(b) of Algorithm 1 only,

$$T^{-1/2} \hat{C}^{(r)} \sum_{i=1}^{\lfloor Tu \rfloor} \varepsilon_{r,i}^* \xrightarrow{w}_p C_0^{(r)} M(u) \, , \quad u \in [0,1], \tag{5.3}$$

where the p-variate stochastic process is given by, $M(\cdot) = \int_0^{\cdot} \sigma(s) \, dB_p(s)$, with B_p a p-dimensional standard Brownian motion, and $C_0^{(r)}$ is again as defined in part (i).

Remark 5 The proof of Proposition 1 exploits the fact that, by Lemma 1(i), for any rank $r \leq r_0$ the bootstrap recursion in (4.1) coincides, in the limit, with the recursion $\Delta X_{r,t}^* = \alpha_0^{(r)} \beta_0^{(r)\prime} X_{r,t-1}^* + \sum_{i=1}^{k-1} \Gamma_{i,0}^{(r)} \Delta X_{t-i}^* + \alpha_0^{(r)} \rho_0^{(r)\prime} D_t + \phi_0^{(r)} d_t + \varepsilon_{r,t}^*$ which, by Lemma 1(ii), satisfies the $I(1,r)$ conditions.

This property implies that the bootstrap sample is asymptotically $I(1)$ with r co-integrating relations, as the results in (5.2) and (5.3) coupled with (5.1) formally establish under Assumptions $\mathcal{V}$' and $\mathcal{V}$", respectively.

Using Lemma 1 and Proposition 1, we now establish the asymptotic behaviour of the bootstrap trace statistic $Q^*_{r,T}$ of Algorithm 1 under either Assumption $\mathcal{V}$' or Assumption $\mathcal{V}$". The stated results hold for any $r \leq r_0$.

Proposition 2 *Let the bootstrap statistic $Q^*_{r,T}$ be generated as in Algorithm 1. Then, under the conditions of Lemma 1, and for any $r \leq r_0$: (i) if Assumption $\mathcal{V}$' holds, then $Q^*_{r,T} \xrightarrow{w}_p \mathsf{tr}(\mathcal{Q}_{r,\infty})$, where $\mathcal{Q}_{r,\infty}$ is as defined in* (3.3), *with this result holding regardless of whether the i.i.d. or wild bootstrap re-sampling design is used in step (iii) of Algorithm 1; and (ii) if Assumption $\mathcal{V}$" holds, then provided the wild bootstrap is employed in step (iii) of Algorithm 1, $Q^*_{r,T} \xrightarrow{w}_p \mathsf{tr}(\mathcal{Q}^H_{r_0,\infty})$ where $\mathcal{Q}^H_{r_0,\infty}$ is as defined below* (3.4).

Remark 6 An immediate consequence of Proposition 2 is that under either Assumption $\mathcal{V}$' or Assumption $\mathcal{V}$", the wild bootstrap test based on $Q^*_{r,T}$ will be asymptotically correctly sized under the null hypothesis ($r = r_0$), and will be consistent for all $r < r_0$. This follows from the results noted in section 3 that $Q_{r_0,T} \xrightarrow{w} \mathsf{tr}(\mathcal{Q}_{r_0,\infty})$ under Assumptions 1 and $\mathcal{V}$' while $Q_{r_0,T} \xrightarrow{w} \mathsf{tr}(\mathcal{Q}^H_{r_0,\infty})$ under Assumptions 1 and $\mathcal{V}$", and that $Q_{r,T}$ diverges at rate T rate when $r < r_0$. In view of this, when the wild bootstrap is employed in step (iii) of Algorithm 1, then $p^*_{r_0,T} \xrightarrow{w} U[0,1]$ and $p^*_{r,T} := 1 - G^*_{r,T}(Q_{r,T}) \xrightarrow{p} 0$, for all $r < r_0$. This result therefore holds for the wild bootstrap under Assumptions $\mathcal{V}$, $\mathcal{V}$' and $\mathcal{V}$". The same result holds for the i.i.d. bootstrap under Assumptions $\mathcal{V}$ and $\mathcal{V}$', but not under $\mathcal{V}$".

We conclude this section by stating the following corollary of Proposition 2 which shows that the bootstrap sequential procedure in Algorithm 2 is consistent. The stated result holds for both the i.i.d. and wild bootstrap-based implementations of Algorithm 2 under Assumption $\mathcal{V}$', but only holds for the wild bootstrap variant under Assumption $\mathcal{V}$".

Corollary 1 *Let $\hat{r}$ denote the estimator of the co-integration rank as obtained in Algorithm 2. Then, under the conditions of Proposition 1: $\lim_{T\to\infty} P(\hat{r} = r) = 0$ for all $r = 0, 1, ..., r_0 - 1$; $\lim_{T\to\infty} P(\hat{r} = r_0)$ $= 1 - \eta \cdot \mathbb{I}(r_0 < p)$, and $\lim_{T\to\infty} \sup_{r\in\{r_0+1,...,p\}} P(\hat{r} = r) \leq \eta$.*

6 Numerical Results

Using Monte Carlo simulation we now turn to an investigation of the finite sample performance of the bootstrap procedures based on restricted estimates of the short-run parameters, as detailed in Algorithms 1 and 2 of Section 4. Results are reported for both the i.i.d. and wild bootstrap versions of the re-sampling scheme in step (iii) of the algorithms. These algorithms are also compared with the corresponding asymptotic procedures of Johansen (1996), and with bootstrap algorithms based on unrestricted estimates of the short-run parameters; see Swensen (2006) for the i.i.d. re-sampling scheme, and Cavaliere *et al.* (2010a, 2010b) for the wild bootstrap re-sampling scheme.

As our simulation DGP we consider the following VAR(2) process of dimension $p = 4$,

$$\Delta X_t = \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \varepsilon_t, \quad t = 1, ..., T \tag{6.1}$$

with ε_t a martingale difference sequence (see below), $X_0 = \Delta X_0 = 0$, and $T \in \{50, 100, 200\}$. The long-run parameter vectors are set to $\beta = (1, 0, 0, 0)'$ and $\alpha = (a, 0, 0, 0)'$ (the case of no co-integration obtains for $a = 0$). Regarding the innovation term, and following van der Weide (2002), and assume that ε_t may be written as the linear map $\varepsilon_t = \Lambda e_t$, where Λ is an invertible $p \times p$ matrix which is constant over time, while the p components of $e_t := (e_{1t}, ..., e_{pt})'$ are independent across $i = 1, ..., p$. In the case where the individual components follow a standard $GARCH(1, 1)$ process (as is the case with Model C below), the process is known as $GO\text{-}GARCH(1, 1)$. Notice that, by definition, the PLR statistic does not depend on the matrix Λ, as the eigenvalue problem in (3.1) has the same eigenvalues upon re-scaling (as can be seen by simply pre- and post-multiplying by Λ^{-1} in (3.1)). This allows us to set $\Lambda = I_p$ in the simulations, with no loss of generality.

In the context of (6.1) we consider for the individual components of e_t the univariate innovation processes and parameter configurations used in Section 4 of Gonçalves and Kilian (2004) and in section 5 of Cavaliere *et al.* (2010b), to which the reader is referred for further discussion. These are as follows:

- **Case A**. e_{it}, $i = 1, ..., p$, is an independent sequence of $N(0, 1)$ variates
- **Case B**. e_{it}, $i = 1, ..., p$, is an independent sequence of Student $t(\nu)$ (normalised to unit variance) variates. Results are reported for $\nu = 5$.
- **Case C**. e_{it} is a standard $GARCH(1, 1)$ process driven by standard normal innovations of the form $e_{it} = h_{it}^{1/2} v_{it}$, $i = 1, ..., p$, where v_{it} is i.i.d. $N(0, 1)$, independent across i, and $h_{it} = \omega + d_0 e_{it-1}^2 + d_1 h_{it-1}$, $t = 0, ..., T$. Results are reported for $d_0 = 0.05$ and $d_1 = 0.94$.
- **Case D**. e_{it} is the first-order AR stochastic volatility [SV] model: $e_{it} = v_{it} \exp(h_{it})$, $h_{it} = \lambda h_{it-1} + 0.5\xi_{it}$, with $(\xi_{it}, v_{it})' \sim$ i.i.d. $N(0, \text{diag}(\sigma_\xi^2, 1))$, independent across $i = 1, ..., p$. Results are reported for $\lambda = 0.951$, $\sigma_\xi = 0.314$.
- **Case E**. e_{it} is a nonstationary, heteroskedastic independent sequence of $N\left(0, \sigma_{it}^2\right)$ variates, where $\sigma_{it}^2 = 1$ for $t \leq \lfloor T\tau \rfloor$ and $\sigma_{it}^2 = \kappa$ for $t > \lfloor T\tau \rfloor$, all $i = 1, ..., p$. Results are reported for $\tau = 2/3$ and $\kappa = 3$ (late positive variance shift).

Notice that cases A and B satisfy Assumption $\mathcal{V}$ (i.i.d. shocks). Under case C, for the chosen parameter configuration, ε_t is globally stationary with finite 4th order moments and, hence, satisfies Assumption $\mathcal{V}$'. Similarly, the SV model of Case D is strictly stationary with bounded 4^{th} order moments, see Carrasco and Chen (1992) and, hence, satisfies Assumption $\mathcal{V}$'. Finally, Case E implies a single, permanent shift in the innovation variance; the resulting error sequence is therefore globally heteroskedastic and satisfies Assumption $\mathcal{V}$".

The reported simulations were programmed using the **rndKMn** pseudo-Gaussian random number generator function of Gauss 9.0. All experiments were conducted using $10,000$ replications. For the bootstrap tests, any replications violating the root check conditions (step (ii) in Algorithms 1 and 2 and step (iii) in Algorithms 1 and 2 of Swensen, 2006) were discarded and the experiment continued until 10,000 valid replications were obtained. For each bootstrap procedure we report the frequency with which such violations occurred.[4] The number of replications used in both the i.i.d. and the wild bootstrap algorithms was set to 399. All tests were conducted at the nominal 0.05 significance level. The VAR model was fitted with a restricted constant when calculating all of the tests. For the standard PLR tests the asymptotic critical values used were taken from Table 15.2 of Johansen (1996).

We first in section 6.1 consider the case of no co-integration, setting $a = 0.0$, and then turn in section 6.2 to the case of a single co-integration vector, setting $a = -0.4$.

6.1 The no co-integration case ($r_0 = 0$)

In the non-co-integrated case, (6.1) reduces to the VAR(1) in first differences, $\Delta X_t = \Gamma\Delta X_{t-1} + \varepsilon_t$, $t = 1, ..., T$. As in Johansen (2002, section 3.1), we set $\Gamma_1 := \gamma I_4$, so that the $I(1,r)$ conditions are met with $r = 0$, provided $|\gamma| < 1$. Results are reported for $\gamma \in \{0.0, 0.5, 0.8, 0.9\}$.

We first consider the size of the asymptotic PLR test and the various bootstrap analogue tests for $r = 0$. The bootstrap tests from section 5 of this paper are denoted by $Q_{0,T}$ (asymptotic test), $Q^{*\text{iid}}_{0,T}$ (Algorithm 1, i.i.d. re-sampling), $Q^{*\text{w}}_{0,T}$ (Algorithm 1, wild re-sampling), while the bootstrap tests based on unrestricted estimation of the short-run parameters, as originally proposed by Swensen (2006) and Cavaliere *et al.* (2010*a*,b) are denoted by $\tilde{Q}^{*,\text{iid}}_{0,T}$ and $\tilde{Q}^{*,\text{w}}_{0,T}$, respectively. Empirical rejection frequencies [ERFs] of these tests for $r = 0$ for case A (i.i.d. Gaussian shocks) and case B (i.i.d. $t\,(5)$ shocks), are reported in Tables 1.1 and 2.1 respectively.

[**TABLES 1.1, 1.2, 2.1, 2.2 ABOUT HERE**]

It is seen from the results in Tables 1.1 and 2.1 that the standard asymptotic test for $r = 0$, $Q_{0,T}$, displays very poor finite sample size control. Even in the simplest case where $\gamma = 0$ and the shocks are Gaussian, the ERF is around 19% for $T = 50$, improving somewhat to around 8% for $T = 200$. However, as γ increases, size control deteriorates markedly; for instance, when $\gamma = 0.9$ the size of the asymptotic test exceeds 93% when $T = 50$, and is still as high as 45% for $T = 200$. In contrast, the ERFs of the bootstrap tests, $Q^{*,\text{iid}}_{0,T}$ and $Q^{*,\text{w}}_{0,T}$, all lie very close to the nominal 5% level. The test based on wild bootstrap re-sampling, $Q^{*,\text{w}}_{0,T}$, appears to be slightly more conservative than its i.i.d. analog, $Q^{*,\text{iid}}_{0,T}$: for Gaussian shocks and $\gamma = 0$, $Q^{*,\text{iid}}_{0,T}$ has size ranging from 4.6% ($T = 50$) to 4.9% ($T = 200$), while the wild bootstrap test $Q^{*,\text{w}}_{0,T}$ has size ranging from 3.2% ($T = 50$) to 4.4% ($T = 200$) in this setting. Interestingly, when $\gamma = 0.9$, $Q^{*,\text{w}}_{0,T}$ controls size extremely well, both

[4]The Gauss procedure for computing the bootstrap algorithms is available from the authors upon request.

under Gaussian and $t(5)$ errors; for example, in the Gaussian case, when $T = 50$, $Q_{0,T}^{*,\mathrm{w}}$ has size 7.1% while $Q_{0,T}^{*,\mathrm{iid}}$ has size around 11%. In line with the results in CRT, and for the simulation DGPs considered here, the bootstrap PLR tests based on Algorithm 1 are seen to be clearly preferable to the corresponding bootstrap PLR tests based on unrestricted estimates of the short run parameters. In the final example above the i.i.d. bootstrap test of Swensen (2006), $\tilde{Q}_{0,T}^{*,\mathrm{iid}}$, has an ERF of around 30%, while the corresponding ERF for the wild bootstrap analogue of Cavaliere *et al.* (2010*a*, 2010*b*), $\tilde{Q}_{0,T}^{*,\mathrm{w}}$, is about 25%. This illustrates the substantial improvements in size control that can be obtained by estimating all the parameters restrictedly, i.e. by imposing the null rank being tested.

The associated results for the sequential procedures are reported in Table 1.2 (Gaussian shocks) and Table 2.2 ($t(5)$ shocks). Since all of the tests were run at the (asymptotic) 5% significance level and the DGP satisfies Assumption $\mathcal{V}$, both the standard asymptotic sequential procedure and all of the bootstrap sequential procedures should (in the limit) select $r = 0$ with probability 95% and $r > 0$ with probability 5%. As with the results in Tables 1.1 and 2.1, among the various algorithms considered, Algorithm 2 of CRT again appears to deliver the best performance in terms of its ability to select the true co-integration rank, $r_0 = 0$. Of the two re-sampling options within step (iii) of this algorithm, the procedure based on i.i.d. re-sampling appears to be slightly more liberal than its wild bootstrap analogue, which is extremely accurate, even for large values of γ.

Finally, comparing the results in Tables 1.1 and 2.1 and the results in Tables 1.2 and 2.2, it is seen that, for a given PLR test and associated sequential procedure, the results appear little affected by whether the shocks are Gaussian or $t\,(5)$ distributed.

[**TABLES 3.1, 3.2, 4.1, 4.2 ABOUT HERE**]

We now consider the corresponding results for the two (stationary) conditionally heteroskastic processes specified in cases C (independent stationary GARCH(1,1) processes) and D (stationary autoregressive stochastic volatility processes) above. Results for the tests of $r = 0$ are reported in Tables 3.1 (case C) and 4.1 (case D). In both cases, the standard asymptotic test is seen to be massively oversized: for example, when the shocks follow a SV process, even for $T = 200$ the size of $Q_{0,T}$ ranges between 26.9% ($\gamma = 0$) and 55.7% ($\gamma = 0.9$). Of the two stationary conditionally heteroskedastic shock processes considered, it is the autoregressive stochastic volatility case, Case D, which has the strongest impact on the size of the asymptotic PLR test, this because the chosen parameter configuration implies relatively strong serial dependence in the conditional variance of the innovations.

In the GARCH(1,1) case, the bootstrap i.i.d. test from Algorithm 1, $Q_{0,T}^{*,\mathrm{iid}}$, displays very accurate size controls: the ERFs associated with this test are little different from those observed in cases A and B (i.i.d. shocks). Overall, the wild bootstrap test of Algorithm 1, $Q_{0,T}^{*,\mathrm{w}}$, although slightly undersized under for small values of γ, does an excellent job, in particular for the larger values of γ considered. It is, however, where the innovation process follows a SV process (case D) that the benefits of the wild bootstrap tests over the other tests become clear. Under case D, the size properties of the i.i.d. bootstrap test, $Q_{0,T}^{*,\mathrm{iid}}$, although representing an improvement over the asymptotic test, are still largely unsatisfactory; for $T = 200$, the ERFs are all still above 17%. Conversely, the ERFs of its wild bootstrap analogue, $Q^{*,\mathrm{w}}_{0,T}$, are all very close to the nominal 5% level, even for small T and large γ.

As was seen for the i.i.d. cases A and B, the bootstrap LR tests $Q^{*,\mathrm{iid}}_{0,T}$ and $Q^{*,\mathrm{w}}_{0,T}$ have much better size than the corresponding bootstrap tests of Swensen (2006), $\tilde{Q}^{*,\mathrm{iid}}_{0,T}$, and Cavaliere *et al.* (2010*a*, 2010*b*), $\tilde{Q}^{*,\mathrm{w}}_{0,T}$. This is particularly in evidence for the larger values of γ considered. For example, in the case of GARCH shocks (SV shocks), $T = 50$ and $\gamma = 0.9$, the wild bootstrap test $\tilde{Q}^{*,\mathrm{w}}_{0,T}$ has size of 26.5% (28.9%), while the test based on the wild bootstrap version of Algorithm 1, $Q^{*,\mathrm{w}}_{0,T}$, has size of 6.7% (7.6%).

Turning to the associated results for the sequential procedures, see Table 3.2 (case C) and Table 4.2 (case D), it is again the version of Algorithm 2 which employs the wild bootstrap re-sampling scheme in step (iii) that has the best available performance for the simulation DGPs considered here in terms of its probability of selecting the true rank $r_0 = 0$. The version of Algorithm based on the i.i.d. re-sampling in step (iii), while performing well under GARCH errors, is misleading in the presence of SV: for example, when $T = 200$ it detects one (or more) co-integration relation at least 16% of the time. Conversely, the wild bootstrap version of Algorithm 2 turns out to perform particularly well for all values of γ considered.

[**TABLES 5.1, 5.2 ABOUT HERE**]

We now turn to the case of non-stationary heteroskedasticity by reporting, in Tables 5.1 and 5.2, the results for the case of a one-time change in volatility occurring in each of the p errors e_{it} (Case E). This process satisfies Assumption $\mathcal{V}$" with $\sigma(\cdot)$ a non-constant step function and, hence, both the asymptotic PLR tests and the bootstrap PLR tests based on i.i.d. re-sampling would be expected to be unreliable; see Cavaliere *et al.* (2010*b*). Conversely, we expect from Proposition 2 above that the bootstrap tests based on wild re-sampling will be approximately correctly sized.

The ERFs reported in Tables 5.1 are indeed in line with our theoretical results. Specifically, under a one-time change in volatility the asymptotic test, $Q_{0,T}$, is extremely unreliable in terms of size. The bootstrap PLR test from Algorithm 1 based on i.i.d. re-sampling, $Q^{*\mathrm{iid}}_{0,T}$, is also unreliable, with size ranging from 25.6% (when $\gamma = 0.9$) to 34% (when $\gamma = 0$) for $T = 200$. Conversely, the size properties of the wild bootstrap PLR test from Algorithm 1, $Q^{*\mathrm{w}}_{0,T}$, seem largely satisfactory. A significant degree of finite sample oversize can, however, occur when $T = 50$, although these distortions substantially reduce as the sample size increases; for example, when $\gamma = 0.9$ ($\gamma = 0$), $Q^{*\mathrm{w}}_{0,T}$ has size of 12.3% (10.1%), reducing to 6.7% (7.1%) for $T = 200$. It is also worth noting that $Q^{*\mathrm{w}}_{0,T}$ performs considerably better than the corresponding wild bootstrap test (based on unrestricted estimation of the short run parameters) proposed in Cavaliere *et al.* (2010*a*, 2010*b*), $\tilde{Q}^{*\mathrm{w}}_{0,T}$. For example, for $T = 50$ the size of $\tilde{Q}^{*\mathrm{w}}_{0,T}$ is about 28.3% (17.9%) when $\gamma = 0.9$ ($\gamma = 0$), while for $T = 200$, the size of $\tilde{Q}^{*\mathrm{w}}_{0,T}$ is about 11.1% (8.7%) when $\gamma = 0.9$ ($\gamma = 0$).

The superiority of the bootstrap procedures based on restricted estimation are further confirmed by the sequential results reported in Table 5.2. Consistent with the results in Table 5.1, the procedure based on the bootstrap $Q^{*\mathrm{w}}_{r,T}$ tests gets considerably closer to selecting the true rank with frequency 95%. Conversely, both the standard procedure based on $Q_{r,T}$ and the bootstrap procedure based on i.i.d. re-sampling of the residuals, perform very poorly. It is also worth noting that, for all the sample sizes considered and for all values of γ considered, the wild bootstrap version of Algorithm 2 again performs substantially better than the sequential procedure outlined in Cavaliere *et al.* (2010*a*, 2010*b*), with the latter tending to detect co-integration too frequently in finite samples.

We conclude this subsection by examining the percentage of times the bootstrap algorithms fail to generate I(1) samples, hence violating the root check conditions (either step (ii) in Algorithm 1 or step (iii) in Algorithm 1 of Swensen, 2006, as appropriate).[5] It can be observed that such percentages are quite low across all models considered. Some violations occur when volatility is persistent (cases D and E) and $T = 50$, but other things being equal the frequency of such failures decreases rapidly as the sample size becomes bigger. Overall, CRT's Algorithm 1 generates fewer explosive samples than the corresponding algorithms in Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*). The only exception is Case E with T small and γ large. As far as the sequential algorithms are considered, CRT's Algorithm 2 never generates more failures of the root check condition than the corresponding algorithms from Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*) and, hence, we believe it is preferable to these procedures not only in terms of its ability to detect the true co-integration rank, but also in terms of how frequently it generates I(1) bootstrap samples.

6.2 The co-integrated case ($r_0 = 1$)

We now consider the VAR(2) in (6.1) with the long-run parameter vectors set to $\beta = (1, 0, 0, 0)'$, $\alpha = (a, 0, 0, 0)'$ and with $a = -0.4$. As in CRT, the lagged differences matrix Γ_1 is specified as

$$\Gamma_1 := \begin{bmatrix} \gamma & \delta & 0 & 0 \\ \delta & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix}$$

with $\gamma = 0.8$ and $\delta \in \{0, 0.1, 0.2, 0.3\}$. For all of these parameter combinations, X_t is $I(1)$ with co-integrating rank $r_0 = 1$. As in CRT, the role of the parameter δ is to isolate the violation or otherwise of the auxiliary conditions given in Assumption 2 of Swensen (2009); in particular, these conditions are satisfied only for $\delta = 0$ or $\delta = 0.1$. As described in CRT, footnote 4, the bootstrap tests for $r = 0$ in Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*) are able to generate I(1,0) bootstrap samples only for $|\delta| < 0.2$. For $\delta > 0$, these bootstrap algorithms generate explosive processes with non-negligible probability, even as T get large.

[5] Since steps (i) and (ii) of Algorithm 1 do not depend on the method used to re-sample the residuals (i.i.d. or wild bootstrap re-sampling), the number of root check violations associated with $Q^{*\mathrm{iid}}_{0,T}$ and $Q^{*\mathrm{w}}_{0,T}$ will be identical and are therefore reported only once. The same equivalence applies to the tests $\tilde{Q}^{*\mathrm{iid}}_{0,T}$ and $\tilde{Q}^{*\mathrm{w}}_{0,T}$ of Swensen (2006) and Cavaliere *et al.* (2010a, 2010b). This feature does not hold for the sequential procedures, however, since algorithms that tend to select higher values of the co-integration rank will necessarily perform a larger number of root checks.

[TABLES 6.1, 6.2, 7.1, 7.2 ABOUT HERE]

For i.i.d. shocks, the ERFs of the tests for $r = 1$ are reported in Tables 6.1 (Gaussian shocks) and 7.1 ($t\,(5)$ shocks). It is seen from these results that the standard asymptotic test for $r = 1$, $Q_{1,T}$, again displays very poor finite sample size control. For instance, for both Gaussian and $t\,(5)$ shocks, the ERFs are around 46% for $T = 50$ and still around 13% for $T = 200$. In contrast, the ERFs of the bootstrap tests, $Q_{1,T}^{*\text{iid}}$ and $Q_{1,T}^{*\text{w}}$, from Algorithm 1 all lie very close to the nominal 5% level, even for $T = 50$ (with $Q_{1,T}^{*\text{w}}$ again being marginally conservative). In line with what was observed for the $r_0 = 0$ case in section 6.1, $Q_{1,T}^{*,\text{iid}}$ and $Q_{1,T}^{*,\text{w}}$ are better sized than the corresponding bootstrap tests of Swensen (2006), $\tilde{Q}_{1,T}^{*,\text{iid}}$, and Cavaliere *et al.* (2010*a*, 2010*b*), $\tilde{Q}_{1,T}^{*,\text{w}}$, both of which are too liberal, with size of around 10% in many cases.

The bootstrap tests obtained from Algorithm 1 also have good properties in terms of empirical power. The ERFs of the i.i.d. variant for $r = 0$, $Q_{0,T}^{*\text{iid}}$, are in most cases not lower than those of Swensen's bootstrap $\tilde{Q}_{0,T}^{*\text{iid}}$ test. The only exceptions occur for $\delta = 0, 0.1$ when $T = 50$ where the empirical power of $Q_{0,T}^{*\text{iid}}$ is slightly inferior to that of $\tilde{Q}_{0,T}^{*\text{iid}}$. However, this is largely an artefact of the severe over-sizing of $\tilde{Q}_{0,T}^{*\text{iid}}$ under the null hypothesis $r = 0$ ($\tilde{Q}_{0,T}^{*\text{iid}}$ has size over 19% when $\delta = 0$ and $r = 0$ for $T = 50$; cf. Tables 1.1 and 2.1 for $\gamma = 0.8$). Turning to the wild bootstrap version of the test, the ERFs of the $Q_{0,T}^{*\text{w}}$ test for $r = 0$, although only slightly lower than the ERFs of the $Q_{0,T}^{*\text{iid}}$ test, are always higher than that of the wild bootstrap $\tilde{Q}_{0,T}^{*\text{w}}$ test of Cavaliere *et al.* (2010*a*, 2010*b*). Similarly, while it might appear on a first reading of the results in Tables 6.1 and 7.1 that the asymptotic $Q_{0,T}$ always displays higher power than the corresponding bootstrap tests, this is again an artefact of its poor size control. To illustrate, for $T = 50$, rank 1 and i.i.d. Gaussian shocks (Table 6.1, entry $\delta = 0$), the ERF of, for example, the wild bootstrap $Q_{0,T}^{*\text{w}}$ test is 50.8%, while the ERF of the asymptotic $Q_{0,T}$ test is 97.6%. However, if we look at the corresponding size results (cf. Table 1.1 for $\gamma = 0.8$) it can be clearly seen that under the null hypothesis of rank zero, while the $Q_{0,T}^{*\text{w}}$ test has an ERF of around 5.5%, very close to the nominal level, the $Q_{0,T}$ test has an ERF of 80.2%, grossly in excess of the nominal level.

Tables 6.2 and 7.2 report the associated results for the sequential procedures. Overall, among the various procedures considered, Algorithm 2 based on i.i.d. re-sampling appears to deliver the best performance in terms of its ability to select the true co-integration rank, $r_0 = 1$, followed by its wild bootstrap version. As expected, Algorithm 2 is not affected by the value of δ, whereas in contrast it is clearly seen that the behaviour of the sequential algorithms of Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*) are strongly affected by the value of δ.

As with the results in section 6.1, it is seen that, for a given PLR test and associated sequential procedure, the finite sample behaviour appears little affected by whether the shocks are Gaussian or $t\,(5)$ distributed.

[TABLES 8.1, 8.2, 9.1, 9.2 ABOUT HERE]

We now turn to cases C (independent stationary GARCH(1,1) processes) and D (stationary stochastic volatility processes). As was also observed in section 6.1 for the non-co-integrated DGP, the standard asymptotic PLR test is massively oversized, in particular under SV where even for $T = 200$, the size of $Q_{1,T}$ is about 25%. While the corresponding i.i.d. bootstrap test from Algorithm 1, $Q^{*,\mathrm{iid}}_{1,T}$, delivers very accurate size control in the GARCH(1,1) case, under case D the size properties of this test deteriorate, although they are still significantly better than those associated to the asymptotic test. Conversely, the ERFs of the wild bootstrap $Q^{*,\mathrm{w}}_{1,T}$ test (which is only marginally undersized under GARCH shocks), are all very close to the nominal 5% level, even for the smaller values of T and for all the values of δ considered. As in the two i.i.d. shock cases, the $Q^{*,\mathrm{iid}}_{1,T}$ and $Q^{*,\mathrm{w}}_{1,T}$ bootstrap tests from Algorithm 1 display superior size control than the corresponding bootstrap tests of Swensen (2006), $\tilde{Q}^{*,\mathrm{iid}}_{1,T}$, and Cavaliere *et al.* (2010*a*, 2010*b*), $\tilde{Q}^{*,\mathrm{w}}_{1,T}$, which are again far too liberal.

In terms of empirical power, it can be seen that the $Q^{*,\mathrm{iid}}_{0,T}$ and $Q^{*,\mathrm{w}}_{0,T}$ tests from Algorithm 1 perform well with respect to their $\tilde{Q}^{*\mathrm{iid}}_{0,T}$ and $\tilde{Q}^{*\mathrm{w}}_{0,T}$ counterparts. There are only a few cases where the latter display higher rejection rates and these occurrences can be explained by the degree of over-sizing present in the latter.

Turning to the sequential procedures, see Table 8.2 (case C) and Table 9.2 (case D), it can be clearly seen that Algorithm 2 delivers the best results overall. As in the $r_0 = 0$ case, the implementation of Algorithm 2 which employs i.i.d. re-sampling, while performing best in the presence of GARCH errors and $T = 50$, appears to be dominated by Algorithm 2 with wild bootstrap re-sampling when $T = 100$ and $T = 200$. In addition, although asymptotically valid, in finite samples i.i.d. re-sampling tends to selection too many co-integrating relations; for example, in the SV case even when $T = 200$, it selects $r = 1$ only around 85% of the time, while its wild bootstrap analogue selects $r = 1$ at least 93% of the time. For all of the parameter configurations considered, the two versions of Algorithm 2 (i.i.d. and wild) of dominate their counterparts in Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*), even for $\delta = 0$ and $\delta = 0.1$.

We now compare, for cases A–D, the frequency with which the bootstrap recursions fail to generate valid $I(1)$ bootstrap samples. Taking the sequential procedures under i.i.d. shocks (cases A–B) to illustrate, the percentage of times Algorithm 2 generates explosive bootstrap samples is remarkably small; in particular, it never exceeds 0.3%, 0.2% and 0.1% for $T = 50$, 100 and 200, respectively. In contrast, Algorithm 2 of Swensen (2006) and the corresponding sequential wild bootstrap algorithm of Cavaliere *et al.* (2010*a*, 2010*b*) display a much higher number of failures of the $I(1, r)$ conditions, even when they are asymptotically valid ($\delta = 0.0$ or $\delta = 0.1$). For instance, when $T = 50$ ($T = 100$) and $\delta = 0.1$, explosive bootstrap samples are generated more than 8% (2%) of the time. For $I(1)$ DGPs with $\delta \geq 0.2$, this failure rate increases substantially; e.g. when $\delta = 0.3$ explosive samples are generated about 50% (92%) of the time for $T = 50$ ($T = 200$).

[**TABLES 10.1, 10.2 ABOUT HERE**]

We conclude this section by looking at the case of non-stationary heteroskedasticity. Tables 10.1 and 10.2 report the results for a common, one-time change in volatility (case E) under rank $r_0 = 1$. As noted in Section 6.1, under Assumption $\mathcal{V}$" with $\sigma(\cdot)$ non-constant, both the asymptotic tests and the bootstrap tests based on i.i.d. re-sampling do not preserve their usual consistency properties while, according to Proposition 2, the bootstrap test from Algorithm 1 with wild re-sampling in step (iii) is expected to be approximately correctly sized.

The results in Table 10.1 clearly show that under a one-time change in volatility both the asymptotic test, $Q_{1,T}$, and bootstrap tests based on i.i.d. re-sampling (both the $Q_{1,T}^{*\text{iid}}$ test and Swensen's i.i.d. $\tilde{Q}_{1,T}^{*\text{iid}}$ test) display highly unreliable size properties; for example, when $T = 200$, $Q_{1,T}^{*}$ has size around 34%, $Q_{1,T}^{*\text{iid}}$ about 18.5% and $\tilde{Q}_{1,T}^{*\text{iid}}$ at least 20%. In contrast, our $Q_{0,T}^{*\text{w}}$ wild bootstrap test, is largely satisfactory; its empirical size is only slightly above 5% for all values of δ. Moreover, $Q_{0,T}^{*\text{w}}$ outperforms the wild bootstrap test of Cavaliere *et al.* (2010*a*, 2010*b*), $\tilde{Q}_{0,T}^{*\text{w}}$.

Again, these findings are further supported by the results for the corresponding sequential procedures reported in Table 10.2. The sequential procedure based on $Q_{r,T}^{*\text{w}}$ selects the true rank with frequency very close to 95% throughout. Conversely, both the standard procedure based on $Q_{r,T}$ and the bootstrap procedure based on i.i.d. re-sampling of the residuals, tend to over-estimate the co-integration rank. Moreover, the wild bootstrap version of Algorithm 2 again outperforms the sequential procedure of Cavaliere *et al.* (2010*a*, 2010*b*), even in cases where the latter is asymptotically valid ($\delta = 0$ and $\delta = 1$).

We conclude by examining the frequencies with which the various bootstrap sequential algorithms violate the root check conditions. As far as Algorithms 1 and 2 are concerned, it can be seen that although some violations do occur for the smallest sample size considered, $T = 50$, the frequency of such failures drops to below 1% for $T = 200$. Conversely, the algorithms in Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*) display a substantial number of failures, even in cases where they are asymptotically valid. Overall, as in the $r_0 = 0$ case, at least for the simulation DGPs considered here, Algorithm 2 based on wild bootstrap re-sampling not only has the best performance in terms of selecting the correct co-integration rank, but also outperforms the other algorithms in terms of its ability to generate I(1) bootstrap samples.

7 Conclusions

In this paper we have discussed the bootstrap implementations of the pseudo likelihood ratio co-integration rank test and associated sequential procedure of Johansen (1996) that have recently been proposed in Cavaliere *et al.* (2012). Unlike the bootstrap procedures of Swensen (2006) and Cavaliere *et al.* (2010*a*, 2010*b*), these are based only on restricted estimates of the underlying VAR model; the other bootstrap procedures mentioned use a mixture of restricted and unrestricted estimates. For the case of an i.i.d. re-sampling scheme, Cavaliere *et al.* (2012) demonstrate that their proposed bootstrap procedure is both asymptotically correctly sized and consistent when the VAR is driven by i.i.d. innovations, without the need for the conditions laid out in Swensen (2009) to hold on the underlying VAR. In contrast the bootstrap procedures of Swensen (2006) and Cavaliere *et al.* (2010a, 2010b) require these conditions to hold. We have extended the work of Cavaliere *et al.* (2012) by analysing how their bootstrap procedures behave when the innovations are heteroskedastic, also proposing a wild bootstrap based implementation of their approach. Precisely, we have shown that, when the wild bootstrap implementation is used, this approach is asymptotically correctly sized and consistent under either conditional or unconditional heteroskedasticity, again without the requirement that the conditions of Swensen (2009) hold. When the i.i.d. bootstrap is used, this outcome no longer holds in the case of unconditional heteroskedasticity but does continue to hold under conditional heteroskedasticity.

A small Monte Carlo experiment suggested that, at least on the basis of the set of simulation DGPs considered, the procedure based on the wild bootstrap works very well in finite samples for a variety of models of heteroskedasticity (both conditional and unconditional), outperforming (at least for the simulation DGPs considered) not only asymptotic-based procedures, but also the corresponding bootstrap procedures from Swensen (2006) and Cavaliere *et al.* (2010a, 2010b). Further numerical investigation of the relative performance of the procedures discussed in this paper for a wider set of simulation DGPs would constitute a useful topic for future research.

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Appendix

A.1 Preliminary Lemma

For the proofs in Appendix A.2 covering the case where innovations satisfy Assumption $\mathcal{V}$" a minor generalization of the law of large numbers (LLN) in Lemma A.1 of Cavaliere *et al.* (2010*b*) is needed. Specifically, Lemma A.1 of Cavaliere *et al.* (2010*b*) holds under Assumption $\mathcal{V}$" but with the additional assumption of symmetry, which we relax now relax.

As in Lemma A.1 of Cavaliere *et al.* (2010*b*) consider the p-dimensional heteroskedastic VAR processes:

$$\begin{aligned} Y_t &= A_1 Y_{t-1} + \ldots + A_m Y_{t-m} + \varepsilon_t, \\ X_t &= B_1 X_{t-1} + \ldots + B_n X_{t-n} + \varepsilon_t, \end{aligned} \tag{A.1}$$

with ε_t satisfying Assumption $\mathcal{V}$", i.e. the ε_t are allowed to be asymmetrically distributed. The characteristic polynomials are denoted as $A(z) = 1 - A_1 z - \ldots - A_m z^m$ and $B(z) = 1 - B_1 z - \ldots - B_n z^n$ respectively for the two autoregressions. The processes are well-defined for $t = 1, .., T$ with fixed initial values $(Y_0', Y_{-1}', ..., Y_{-m+1}')'$ and $(X_0', X_{-1}', ..., X_{-n+1}')'$.

Lemma A.1" *Consider the VAR heteroskedastic processes Y_t and X_t defined in (A.1), where the roots of* $\det|A(z)| = 0$ *and* $\det|B(z)| = 0$ *are all assumed to lie outside the unit circle. Then as $T \to \infty$, for $k \geq 0$,*

$$\frac{1}{T}\sum_{t=1}^{T} Y_t X_{t+k}' \xrightarrow{p} \sum_{i=0}^{\infty} \Theta_i \Sigma \Gamma_{i+k}' \tag{A.2}$$

where $\Sigma := \int_0^1 \Sigma(s)\,ds$, *and* Θ_i *and* Γ_i *are the coefficients obtained by inversion of the characteristic polynomials* $A(z)$ *and* $B(z)$ *respectively.*

Proof. Rewrite (A.2) as

$$\frac{1}{T}\sum_{t=1}^{T} Y_t X'_{t+k} = \frac{1}{T}\sum_{t=1}^{T} E\left(Y_t X'_{t+k}\right) + \frac{1}{T}\sum_{t=1}^{T}\left(Y_t X'_{t+k} - E\left(Y_t X'_{t+k}\right)\right). \tag{A.3}$$

First, the first term on the right hand side of (A.3) converges to $\sum_{i=0}^{\infty}\Theta_i\Sigma\Gamma'_{i+k}$, as shown in Cavaliere *et al.* (2010*b*, proof of Lemma A.1). To show that the second term converges in probability to zero it suffices to establish that, as in the proof of Lemma A.1 of Cavaliere *et al.* (2010*b*):

$$V_T := \frac{1}{T}\sum_{t=1}^{T}\left(U_t U'_t - E\left(U_t U'_t\right)\right) \xrightarrow{p} 0 ,$$

where U_t follows the multivariate vector autoregression, $U_t = AU_{t-1} + \varepsilon_t$, with $\rho(A) < 1$. Without loss of generality we may set $U_0 = 0$ in what follows. Recalling the notation $\Sigma_t = \sigma_t\sigma'_t$, first rewrite V_T as,

$$V_T = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=0}^{t-1} A^i\left(\varepsilon_{t-i}\varepsilon'_{t-i} - \Sigma_{t-i}\right)A^{i\prime} + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=0}^{t-1}\sum_{\substack{j=0\\ j\neq i}}^{t-1} A^i\varepsilon_{t-i}\varepsilon'_{t-j}\left(A^j\right)' =: K_T + J_T.$$

Consider first K_T. Here we have that

$$\begin{aligned}
\|K_T\| &= \left\|\frac{1}{T}\sum_{t=1}^{T}\sum_{i=0}^{t-1} A^i\left(\varepsilon_{t-i}\varepsilon'_{t-i} - \Sigma_{t-i}\right)A^{i\prime}\right\| = \left\|\sum_{i=0}^{T-1} A^i\left(\frac{1}{T}\sum_{t=1}^{T-i}\left(\varepsilon_{t-i}\varepsilon'_{t-i} - \Sigma_{t-i}\right)\right)A^{i\prime}\right\| \\
&\leq \sum_{i=0}^{T-1}\left\|A^i\right\|^2\left\|\frac{1}{T}\sum_{t=1}^{T-i}\left(\varepsilon_{t-i}\varepsilon'_{t-i} - \Sigma_{t-i}\right)\right\|
\end{aligned}$$

and, hence, $K_T \xrightarrow{p} 0$. To see this use the result that $\sum_{i=0}^{T-1}\left\|A^i\right\|^2 \leq \mathcal{C} < \infty$, which obtains by virtue of the fact that $\rho(A) < 1$. Moreover we have,

$$E\left\|\frac{1}{T}\sum_{t=1}^{T-i}\left(\varepsilon_{t-i}\varepsilon'_{t-i} - \Sigma_{t-i}\right)\right\|^2 \leq \frac{1}{T^2}\sum_{t=1}^{T} E\left\|\left(\varepsilon_{t-i}\varepsilon'_{t-i} - \Sigma_{t-i}\right)\right\|^2 = O\left(\frac{1}{T}\right)$$

since for all t, σ_t and Σ_t are finite (see Cavaliere *et al.*, 2010*b*) and ε_t has finite fourth moment.

Turning next to J_T, set $\tilde{\varepsilon}_t := 1(t \geq 1)\,\varepsilon_t$ and rewrite J_T as follows,

$$\begin{aligned}
J_T &= \frac{1}{T}\sum_{t=1}^{T}\sum_{i=0}^{t-1}\sum_{\substack{j=0\\ j\neq i}}^{t-1} A^i\varepsilon_{t-i}\varepsilon'_{t-j}A^{j\prime} = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=0}^{T-1}\sum_{\substack{j=0\\ j\neq i}}^{T-1} A^i\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}'_{t-j}A^{j\prime} \\
&= \sum_{i=0}^{T-1}\sum_{\substack{j=0\\ j\neq i}}^{T-1} A^i\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}'_{t-j}\right)A^{j\prime}.
\end{aligned}$$

This implies that

$$E\left\|J_T\right\| \leq \sum_{i=0}^{T-1}\sum_{\substack{j=0\\ j\neq i}}^{T-1}\left\|A^i\right\|\left\|A^j\right\| E\left\|\frac{1}{T}\sum_{t=1}^{T}\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j}'\right\|.$$

As before, $\sum_{i=0}^{T-1}\sum_{\substack{j=0\\ j\neq i}}^{T-1}\left\|A^i\right\|\left\|A^j\right\| \leq \mathcal{C} < \infty$ and so to show that $J_T \xrightarrow{p} 0$ we need to find an $o(1)$ upper bound for $E\left\|T^{-1}\sum_{t=1}^{T}\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j}'\right\|$ which is independent of i, j. To that end, observe that

$$\left(E\left\|\frac{1}{T}\sum_{t=1}^{T}\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j}'\right\|\right)^2 \leq E\left(\left\|\frac{1}{T}\sum_{t=1}^{T}\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j}'\right\|^2\right) = T^{-2}\sum_{s=1}^{T}\sum_{t=1}^{T}E\left(\tilde{\varepsilon}_{t-i}'\tilde{\varepsilon}_{s-i}\tilde{\varepsilon}_{t-j}'\tilde{\varepsilon}_{s-j}\right).$$

Since ε_t (and hence $\tilde{\varepsilon}_t$) are independent, $E\left(\tilde{\varepsilon}_{t-i}'\tilde{\varepsilon}_{s-i}\tilde{\varepsilon}_{t-j}'\tilde{\varepsilon}_{s-j}\right) \neq 0$ for $t = s$ only, which implies

$$\left(E\left\|\frac{1}{T}\sum_{t=1}^{T}\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j}'\right\|\right)^2 \leq \frac{1}{T^2}\sum_{t=1}^{T}E\left(\tilde{\varepsilon}_{t-i}'\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j}'\tilde{\varepsilon}_{t-j}\right) = \frac{1}{T^2}\sum_{t=1}^{T}E\left\|\tilde{\varepsilon}_{t-i}^2\right\|E\left\|\tilde{\varepsilon}_{t-j}^2\right\| = O\left(\frac{1}{T}\right),$$

as required. □

A.2 Proofs of Lemma 1, Propositions 1-2 and Corollary 1

Proof of Lemma 1. The results in Lemma 1 are proved under Assumption $\mathcal{V}$ (i.i.d. innovations) in Lemma 1 of CRT. We therefore state here the modifications needed to the proof of Lemma 1 of CRT to allow for the case where the innovations follow either Assumption $\mathcal{V}$' or $\mathcal{V}$". In order to do so we will apply results from Cavaliere *et al.* (2010*a*) and Cavaliere *et al.* (2010*b*) for Assumptions $\mathcal{V}$' and $\mathcal{V}$", respectively. The proof of Lemma 1 is given for case (i) of no deterministics; the generalisations to the cases of (ii) a restricted constant and (iii) a restricted linear trend, mimic the arguments in the proof of Lemma 1 of CRT and are therefore omitted in the interests of brevity. We establish in turn that the stated results hold under Assumptions $\mathcal{V}$' and $\mathcal{V}$".

Assumption $\mathcal{V}$':

We start by introducing a convenient normalisation of the co-integration parameters, which allows us to prove part (i) of the Lemma; that is, convergence to the pseudo-true parameter vector $\theta_0^{(r)}$. We then prove part (ii) by establishing that $\theta_0^{(r)}$ does indeed satisfy the $I(1, r)$ conditions for $r \leq r_0$.

Normalisation. Denote by α_0, β_0, $\Psi_0 := (\Gamma_{1,0}, ..., \Gamma_{k-1,0})$ and Σ_0 the true parameters in (2.1). By Lemmas A.1-A.3 of Cavaliere *et al.* (2010*a*), the results of Theorem 11.1 in Johansen (1996) apply. Therefore, in particular, the r_0 largest sample eigenvalues $(\hat{\lambda}_i)_{i=1,...,r_0}$ from (3.1) satisfy, as $T \to \infty$, the population eigenvalue problem, $\left|\lambda\Sigma_{\beta\beta} - \Sigma_{\beta 0}\Sigma_{00}^{-1}\Sigma_{0\beta}\right| = 0$. Here $\Sigma_{ij} := \Omega_{ij} - \Omega_{i2}\Omega_{22}^{-1}\Omega_{2j}$ for $i, j = 0, \beta$ and $\Omega_{ij} := plim_{T\to\infty}\frac{1}{T}\sum Z_{it}Z_{jt}'$ for $i, j = 0, 2, \beta$ with $Z_{\beta t} := \beta' X_t$, $Z_{0t} := \Delta X_t$ and $Z_{2t} := \left(\Delta X_{t-1}', ..., \Delta X_{t-k+1}'\right)'$. As in Lemma 1 of CRT, let $\kappa := (\kappa_1, ..., \kappa_{r_0})$ denote the eigenvectors corresponding to the eigenvalues $\lambda_1 > \lambda_2 > ... > \lambda_{r_0} > 0$, such that $\kappa'\Sigma_{\beta\beta}\kappa = I_{r_0}$. We can then define $\beta_0 := \beta\kappa$ and $\alpha_0 := \alpha(\kappa')^{-1}$. Observe that, $\alpha\beta' = \alpha_0\beta_0'$, while also

$$\Sigma_{\beta_0\beta_0} = I_{r_0} \quad \text{and} \quad \Sigma_{\beta_0 0}\Sigma_{00}^{-1}\Sigma_{0\beta_0} = \text{diag}\left(\lambda_1, ..., \lambda_{r_0}\right), \tag{A.4}$$

with $\Sigma_{\beta_0\beta_0}$and $\Sigma_{\beta_0 0}$ defined as above. Indeed, the relations in (A.4) are the population equivalents of the sample normalisations in (3.5).

Convergence to pseudo-true values. With $\hat{\beta} = \hat{\beta}^{(r_0)}$ the PMLE under the true rank r_0, satisfies by the results in Lemmas A.1-A.3 of Cavaliere *et al.* (2010*a*), $\bar{\beta}_0'(\hat{\beta} - \beta_0) \xrightarrow{p} 0$ and $T^{1/2}\beta_{0\perp}'(\hat{\beta} - \beta_0) \xrightarrow{p} 0$. Next, as in the proof of Lemma 1 of CRT, by Assumption 2, this implies directly that,

$$\bar{\beta}_0'(\hat{\beta}^{(r)} - \beta_0^{(r)}) \xrightarrow{p} 0 \quad \text{and} \quad T^{1/2}\beta_{0\perp}'(\hat{\beta}^{(r)} - \beta_0^{(r)}) \xrightarrow{p} 0 \tag{A.5}$$

where $\beta_0^{(r)} := \beta_0 K_{r_0}^{(r)}$, with $K_{\cdot}^{(r)}$ defined after equation (3.5). Likewise,

$$\hat{\alpha}^{(r)} \xrightarrow{p} \Sigma_{0\beta_0} K_{r_0}^{(r)} = \alpha_0 K_{r_0}^{(r)} =: \alpha_0^{(r)}. \tag{A.6}$$

Regarding $\hat{\Psi}^{(r)}$, define $M_{ij} := \frac{1}{T}\sum_{t=1}^{T} Z_{it}Z_{jt}'$, such that $\hat{\Psi}^{(r)} = \left(M_{02} - \hat{\alpha}^{(r)}\hat{\beta}^{(r)\prime}M_{12}\right)M_{22}^{-1}$, and hence,

$$\hat{\Psi}^{(r)} \xrightarrow{p} \left(\Omega_{02} - \alpha_0 K_{r_0}^{(r)} K_{r_0}^{(r)\prime}\Omega_{\beta_0 2}\right)\Omega_{22}^{-1} = \Psi_0 + \alpha_0 K_{r_0,\perp}^{(r)} K_{r_0,\perp}^{(r)\prime}\Omega_{\beta_0 2}\Omega_{22}^{-1} =: \Psi_0^{(r)} \tag{A.7}$$

with $K_{r_0,\perp}^{(r)} := (0, I_{r_0 - r})'$. Finally, for $\hat{\Sigma}^{(r)}$ it holds that

$$\hat{\Sigma}^{(r)} = S_{00} - \hat{\alpha}^{(r)}\hat{\alpha}^{(r)\prime} \xrightarrow{p} \Sigma_0^{(r)} := \Sigma_0 + \alpha_0 K_{r_0,\perp}^{(r)} K_{r_0,\perp}^{(r)\prime}\alpha_0' > 0. \tag{A.8}$$

Pseudo true values satisfy the I(1,r) conditions, with $r \leq r_0$. Rewrite the DGP as

$$\Delta X_t = \alpha_0^{(r)}\beta_0^{(r)\prime}X_{t-1} + \Psi_0^{(r)}Z_{2t} + \varepsilon_{r,t}, \tag{A.9}$$

see (A.6)–(A.7), with $\varepsilon_{r,t}$ given by

$$\varepsilon_{r,t} = \varepsilon_t + \alpha_0 K_{r_0,\perp}^{(r)} K_{r_0,\perp}^{(r)\prime}\left(\beta_0' X_{t-1} - \Omega_{\beta_0 2}\Omega_{22}^{-1}Z_{2t}\ \right). \tag{A.10}$$

Observe that under Assumption $\mathcal{V}$', $\beta_0^{(r)\prime}X_{t-1}$ and Z_{2t} in (A.9) are uncorrelated with $\varepsilon_{r,t}$. Proceed next as in the proof of Lemma 1 of CRT, by first writing the system in companion form and using identical arguments to see that the I(1,r) conditions hold under $\mathcal{V}$'.

Assumption $\mathcal{V}$*":*

Under Assumption $\mathcal{V}$", we can apply results from Cavaliere *et al.* (2010*b*) who repeatedly apply the LLN for heteroskedastic vector autoregressions; see Cavaliere *et al.* (2010*b*), Lemma A.1. As noted in Appendix A.1 we need the modified version stated in Lemma A.1" to allow for asymmetric innovations. Using Lemma A.1" allows us in particular to replace Σ in the foregoing proof under Assumption $\mathcal{V}$', by $\Sigma := \int_0^1 \sigma(s)\,\sigma(s)'\,ds$. Moreover, note that whenever arguments here and in Cavaliere *et al.*(2010*b*) refer to Lemma A.1, this should subsequently be replaced by a reference to Lemma A.1" above.

Normalisation and convergence to pseudo-true values. Lemmas A.2-A.3 of Cavaliere *et al.* (2010*b*) imply that we get identical results to those given above in the proof under Assumption $\mathcal{V}$' in terms of $\Sigma_{ij} := \Omega_{ij} - \Omega_{i2}\Omega_{22}^{-1}\Omega_{2j}$ for $i,j = 0,\beta$ and Ω_{ij}, $i,j = 0,2,\beta$, but where the limiting expressions for the Ω_{ij} are now as defined in the proof of Lemma A.2 of Cavaliere *et al.* (2010*b*) with $\bar{\Sigma} = \Sigma := \int_0^1 \sigma(s)\sigma(s)' ds$, and with a corresponding change in the definition of $\Sigma_0^{(r)}$.

Pseudo true values satisfy the I(1,r) conditions, with $r \leq r_0$. Rewrite the DGP as in (A.9) and define $\varepsilon_{r,t}$ as in (A.10), then again we note that under Assumption $\mathcal{V}$", $\beta_0^{(r)\prime} X_{t-1}$ and Z_{2t} in (A.9) are uncorrelated with $\varepsilon_{r,t}$. Proceed as in the proof of Lemma 1 of CRT by writing the system in companion form and using identical arguments to see that the I(1,r) conditions hold under $\mathcal{V}$". □

PROOF OF PROPOSITION 1: For the i.i.d. bootstrap, the results in (5.1) and (5.2) are established in Proposition 1 of CRT under Assumption $\mathcal{V}$ (i.i.d. innovations). As in the proof of Lemma 1, we state below the modifications needed for the extensions required to cover Assumptions $\mathcal{V}$' and $\mathcal{V}$", applying results from Cavaliere *et al.* (2010*a*) and Cavaliere *et al.* (2010*b*), respectively, for both the i.i.d. bootstrap and the wild bootstrap. Again we focus on the case *(i)* of no deterministics as the extension to deterministics simply mimics the same arguments as in the proof of Proposition 1 of CRT.

Assumption $\mathcal{V}$':
For $r = r_0$ the results in (5.1) and (5.2) are established in Lemma A.4 of Cavaliere *et al.* (2010*a*). Next, for $r < r_0$, the results hold by the proof of Proposition 1 in CRT, re-defining $\mathbb{X}_t^* = \left(X_{r,t}^{*\prime}, ..., X_{r,t-k+1}^{*\prime}\right)'$, $\Delta\mathbb{X}_{2t} = Z_{2t}$ and $\Upsilon_{ij} = \Omega_{ij}$ for $i, j = 0, 2, \beta, \beta_0$ in terms of the notation introduced above in the proof of Lemma 1 under Assumption $\mathcal{V}$'. Thus the algebraic arguments in CRT using the companion form in terms of $\mathbb{X}_t^*$, see equation (A.9) of CRT, directly yield the representation,

$$X_{r,t}^* = \hat{C}^{(r)} \sum_{i=1}^{t} \varepsilon_{r,t}^* + S_{r,t} T^{1/2}, \tag{A.11}$$

where $\hat{C}^{(r)}$ is as defined in Proposition 1, $\varepsilon_{r,t}^*$ is as defined in Algorithm 1, and $S_{r,t}$ is as defined in the proof of Proposition 1 in CRT. That $P^*(\max_{t=1,...,T} \|S_{r,t}\| > \eta) = o_p(1)$ holds by the arguments given in the proof of Proposition 1 of CRT, using the consistency of the estimators established here in Lemma 1 under Assumption $\mathcal{V}$', provided that $P^*\left(T^{-1/2}\max_{t=1,...,T} \left\|\varepsilon_{r,t}^*\right\| > \eta\right) = o_p(1)$. For the i.i.d. bootstrap, this holds, as in CRT, by applying Chebychev's inequality, the fact that $E^*\left(\varepsilon_{r,t}^{*\prime}\varepsilon_{r,t}^*\right)^2 = T^{-1}\sum_{t=1}^T \left(\hat{\varepsilon}_{r,t}'\hat{\varepsilon}_{r,t} - \bar{\varepsilon}_r'\bar{\varepsilon}_r\right)^2$ and that by Assumption $\mathcal{V}$', ε_t has bounded fourth order moment. For the wild bootstrap, the proof is identical except that, with $\bar{\varepsilon}_r := T^{-1}\sum_{t=1}^T \hat{\varepsilon}_{r,t}$, $E^*\left(\varepsilon_{r,t}^{*\prime}\varepsilon_{r,t}^*\right)^2 = \left[(\hat{\varepsilon}_{r,t} - \bar{\varepsilon}_r)'(\hat{\varepsilon}_{r,t} - \bar{\varepsilon}_r)\right]^2 = \left(\hat{\varepsilon}_{r,t}'\hat{\varepsilon}_{r,t} - \bar{\varepsilon}_r'\bar{\varepsilon}_r\right)^2$.

Next, the result that $T^{-1/2}X_{r,\lfloor Tu\rfloor}^* \xrightarrow{w}_p C_0^{(r)}W(u)$, follows by the consistency in Lemma 1 under Assumption $\mathcal{V}$' together with the convergence result $T^{-1/2}\sum_{t=1}^{\lfloor T\cdot\rfloor} \varepsilon_{r,t}^* \xrightarrow{w}_p W(\cdot)$, which, as in Cavaliere *et al.* (2010*a*), for the wild bootstrap is implied by the pointwise convergence

$$\frac{1}{T}\sum_{t=1}^{\lfloor Tu\rfloor} \hat{\varepsilon}_{r,t}\hat{\varepsilon}_{r,t}' = \frac{1}{T}\sum_{t=1}^{\lfloor Tu\rfloor} \varepsilon_{r,t}\varepsilon_{r,t} + o_p(1) \xrightarrow{p} u\Sigma_0^{(r)} \tag{A.12}$$

while, for the i.i.d. bootstrap, follows by noticing that the bootstrap FCLT in Swensen (2006) holds under Assumption $\mathcal{V}$' as well.

Assumption $\mathcal{V}$":

Similarly to what was established under Assumption $\mathcal{V}$', for $r = r_0$ the results in (5.1) and (5.3) are established in Lemmas A.4 and A.5 of Cavaliere *et al.* (2010*b*). Likewise, for $r < r_0$ the algebraic arguments given in the proof of Proposition 1 of CRT directly yield the representation in (A.11) under Assumption $\mathcal{V}$". Also, that $P^* \left(\max_{t=1,\dots,T} \|S_{r,t}\| > \eta\right) = o_p(1)$ holds by the arguments in CRT, proof of Proposition 1, using the consistency of the estimators established here in Lemma 1 under Assumption $\mathcal{V}$", and using the result that $P^* \left(T^{-1/2} \max_{t=1,\dots,T} \left\|\varepsilon^*_{r,t}\right\| > \eta\right) = o_p(1)$. The latter holds as in the proof of Lemma A.4 in Cavaliere *et al.* (2010*b*), using the fact that under Assumption $\mathcal{V}$" ε_t has bounded fourth order moment.

Next, the result that $T^{-1/2} X^*_{r,\lfloor Tu \rfloor} \xrightarrow{w}_p C_0^{(r)} M(u)$, follows by the consistency in Lemma 1 under Assumption $\mathcal{V}$" together with the convergence result $T^{-1/2} \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon^*_{r,t} \xrightarrow{w}_p M(\cdot)$, which again holds by Cavaliere *et al.* (2010*b*), Lemma A.5. □

Proof of Proposition 2: Under Assumption $\mathcal{V}$' and as in the proof of Theorem 3 of Cavaliere *et al.* (2010*a*), this follows immediately by the results in Proposition 1 using standard arguments and defining $B_{p-r} := \left(\alpha_{0\perp}^{(r)\prime} \Sigma_0^{(r)} \alpha_{0\perp}^{(r)}\right)^{-1/2} \alpha_{0\perp}^{(r)\prime} W$. Likewise, under Assumption $\mathcal{V}$", this holds by Proposition 1 and the proof of Theorem 3 in Cavaliere *et al.* (2010*b*), defining $\tilde{M}_{p-r} := \left(\alpha_{0\perp}^{(r)\prime} \Sigma_0^{(r)} \alpha_{0\perp}\right)^{-1/2} \alpha_{0\perp}^{(r)\prime} M$. □

Proof of Corollary 1. Straightforward and therefore omitted in the interests of brevity. □

TABLE 1.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 0$, IID Gaussian errors

		$Q_{0,T}$	$\tilde{Q}^{*\mathrm{iid}}_{0,T}$	$\tilde{Q}^{*\mathrm{w}}_{0,T}$		$Q^{*\mathrm{iid}}_{0,T}$	$Q^{*\mathrm{w}}_{0,T}$	
γ	T	ERF	ERF	ERF	RC	ERF	ERF	RC
0.0	50	19.1	4.6	4.2	0.0	4.6	3.2	0.0
	100	10.4	4.7	4.7	0.0	4.7	4.1	0.0
	200	7.9	5.0	4.8	0.0	4.9	4.4	0.0
0.5	50	39.1	7.5	6.8	0.0	5.4	3.8	0.0
	100	19.0	6.0	6.0	0.0	5.3	4.6	0.0
	200	11.0	4.9	5.0	0.0	5.0	4.7	0.0
0.8	50	80.2	19.4	16.9	0.4	8.1	5.5	0.0
	100	47.1	11.7	10.8	0.0	6.5	5.2	0.0
	200	23.1	7.1	6.7	0.0	5.6	5.0	0.0
0.9	50	93.3	30.2	25.0	1.9	11.0	7.1	1.6
	100	75.9	22.1	20.3	0.2	8.4	6.9	0.0
	200	44.8	11.9	11.3	0.0	6.3	5.9	0.0

Notes: 'ERF' denotes the empirical rejection rates; 'RC' denotes the percentage of times the bootstrap algorithm generates explosive samples.

TABLE 1.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 0$, IID Gaussian errors

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
γ	T	$r=0$	1	2	$3,4$	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC
0.0	50	80.9	16.4	2.3	0.4	95.4	4.1	0.5	0.0	0.0	95.8	3.9	0.3	0.0	0.0	95.5	4.2	0.3	0.0	0.0	96.8	2.9	0.3	0.0	0.0
	100	89.6	9.1	1.1	0.2	95.3	4.3	0.4	0.0	0.0	95.3	4.2	0.5	0.0	0.0	95.3	4.3	0.4	0.0	0.0	95.9	3.5	0.5	0.1	0.0
	200	92.1	7.2	0.6	0.1	95.1	4.6	0.3	0.1	0.0	95.2	4.4	0.3	0.0	0.0	95.1	4.5	0.3	0.1	0.0	95.6	4.0	0.4	0.1	0.0
0.5	50	60.9	29.2	7.7	2.1	92.5	6.6	0.8	0.2	0.1	93.2	6.0	0.8	0.1	0.1	94.6	4.8	0.5	0.1	0.0	96.2	3.2	0.4	0.2	0.1
	100	81.0	16.0	2.4	0.6	94.0	5.2	0.7	0.1	0.0	94.0	5.4	0.5	0.1	0.0	94.7	4.7	0.6	0.1	0.0	95.4	4.0	0.5	0.2	0.0
	200	89.0	9.9	1.0	0.2	95.1	4.5	0.4	0.1	0.0	95.0	4.6	0.3	0.0	0.0	95.0	4.5	0.4	0.1	0.0	95.3	4.2	0.4	0.1	0.0
0.8	50	19.8	38.2	25.5	16.5	80.6	16.2	2.6	0.5	2.5	83.2	13.8	2.7	0.4	2.1	91.9	7.2	0.7	0.2	0.3	94.5	4.5	0.6	0.4	0.2
	100	52.9	34.0	9.6	3.4	88.4	9.9	1.4	0.4	0.2	89.2	9.3	1.2	0.3	0.2	93.5	5.8	0.6	0.1	0.1	94.8	4.4	0.6	0.2	0.1
	200	76.9	19.4	3.1	0.6	92.9	6.4	0.6	0.1	0.0	93.3	6.0	0.6	0.1	0.0	94.4	5.1	0.5	0.1	0.0	95.0	4.5	0.4	0.2	0.0
0.9	50	6.7	23.6	30.3	39.4	69.8	23.7	5.4	1.1	9.2	75.0	19.0	4.7	1.2	8.4	89.0	9.6	1.2	0.2	2.1	92.9	5.8	0.8	0.5	2.2
	100	24.1	38.0	23.3	14.7	77.9	18.2	3.2	0.6	1.7	79.7	17.0	2.7	0.7	1.7	91.6	7.4	0.9	0.1	0.2	93.1	5.8	0.7	0.4	0.2
	200	55.2	32.2	9.7	3.0	88.1	10.5	1.3	0.2	0.3	88.7	9.8	1.3	0.2	0.2	93.7	5.8	0.5	0.1	0.1	94.1	5.2	0.5	0.2	0.1

Notes: 'Restricted' denotes Algorithm 2 of Section 3, 'Unrestricted' denotes Algorithm 2 of Swensen (2006) [IID bootstrap] and Cavaliere et al. (2010a,b) [Wild Bootstrap]. Entries denote the frequency with which each value of r is selected by the given algorithm.

TABLE 2.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 0$, IID $t(5)$ errors

		$Q_{0,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$	
γ	T	ERF	ERF	ERF	RC	ERF	ERF	RC
0.0	50	19.0	5.1	4.1	0.0	5.1	2.8	0.0
	100	11.8	5.4	5.1	0.0	5.5	4.5	0.0
	200	7.9	4.9	4.7	0.0	5.0	4.4	0.0
0.5	50	39.5	7.7	6.8	0.0	5.8	3.6	0.0
	100	19.7	6.0	5.9	0.0	5.4	4.7	0.0
	200	11.4	5.4	5.1	0.0	5.1	4.8	0.0
0.8	50	79.5	19.1	16.5	0.5	8.0	5.0	0.1
	100	46.8	10.6	10.5	0.0	6.0	5.0	0.0
	200	23.7	7.3	7.0	0.0	5.7	5.1	0.0
0.9	50	93.2	29.7	24.9	2.2	10.5	6.4	1.5
	100	77.0	20.9	19.1	0.2	7.9	6.1	0.0
	200	45.4	12.4	11.8	0.0	6.7	5.8	0.0

Notes: see Table 1.1

TABLE 2.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 0$, IID $t(5)$ errors

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
γ	T	$r = 0$	1	2	3, 4	$r = 0$	1	2	3, 4	*RC*	$r = 0$	1	2	3, 4	*RC*	$r = 0$	1	2	3, 4	*RC*	$r = 0$	1	2	3, 4	*RC*
0.0	50	81.0	16.3	2.3	0.5	95.0	4.6	0.4	0.0	0.0	95.9	3.6	0.4	0.1	0.0	94.9	4.8	0.3	0.0	0.0	97.2	2.4	0.3	0.1	0.0
	100	88.2	10.7	0.8	0.3	94.6	4.8	0.5	0.1	0.0	94.9	4.5	0.4	0.1	0.0	94.5	4.9	0.5	0.1	0.0	95.5	3.9	0.4	0.2	0.0
	200	92.1	7.2	0.6	0.1	95.1	4.4	0.4	0.1	0.0	95.3	4.3	0.4	0.0	0.0	95.0	4.5	0.4	0.1	0.0	95.6	3.9	0.4	0.1	0.0
0.5	50	60.6	29.7	7.6	2.3	92.3	6.9	0.7	0.1	0.1	93.2	6.1	0.7	0.1	0.1	94.3	5.3	0.4	0.0	0.0	96.4	3.1	0.4	0.1	0.0
	100	80.3	16.8	2.4	0.5	94.0	5.5	0.5	0.1	0.0	94.1	5.2	0.6	0.1	0.0	94.6	5.0	0.4	0.1	0.0	95.3	4.0	0.5	0.1	0.0
	200	88.6	10.1	1.1	0.2	94.6	4.9	0.5	0.1	0.0	94.9	4.7	0.4	0.1	0.0	95.0	4.6	0.4	0.1	0.0	95.2	4.3	0.4	0.2	0.0
0.8	50	20.5	36.9	25.8	16.7	80.9	16.1	2.4	0.6	2.3	83.5	13.5	2.5	0.5	2.3	92.0	7.1	0.8	0.2	0.2	95.0	4.2	0.6	0.2	0.2
	100	53.2	33.7	9.9	3.3	89.4	9.5	1.0	0.1	0.2	89.5	9.0	1.2	0.2	0.1	94.0	5.4	0.5	0.1	0.0	95.0	4.2	0.6	0.2	0.0
	200	76.3	19.7	3.4	0.7	92.7	6.5	0.8	0.1	0.0	93.0	6.2	0.6	0.2	0.0	94.3	5.2	0.5	0.1	0.0	94.9	4.4	0.5	0.2	0.0
0.9	50	6.8	24.6	29.6	39.1	70.3	23.7	4.8	1.2	8.4	75.1	18.7	4.9	1.3	7.9	89.5	9.3	1.0	0.2	1.9	93.6	5.2	0.8	0.4	1.7
	100	23.0	39.1	24.1	13.9	79.1	17.5	2.9	0.5	1.9	80.9	15.5	3.1	0.6	1.8	92.1	7.1	0.7	0.1	0.1	93.9	5.0	0.8	0.3	0.3
	200	54.6	32.6	9.5	3.2	87.6	10.9	1.3	0.2	0.3	88.2	10.3	1.4	0.1	0.3	93.3	6.1	0.6	0.1	0.0	94.2	5.0	0.6	0.2	0.0

Notes: see Table 1.2

TABLE 3.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 0$, GARCH errors

		$Q_{0,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$	
γ	T	ERF	ERF	ERF	RC	ERF	ERF	RC
0.0	50	18.9	5.0	4.3	0.0	5.2	3.3	0.0
	100	11.1	5.9	5.1	0.0	5.8	4.4	0.0
	200	8.7	6.0	4.7	0.0	5.8	4.6	0.0
0.5	50	39.7	7.7	7.0	0.0	5.7	4.0	0.0
	100	19.9	7.1	6.3	0.0	6.2	4.7	0.0
	200	11.6	6.0	5.1	0.0	5.7	4.6	0.0
0.8	50	80.0	19.5	17.1	0.4	8.2	5.2	0.1
	100	47.5	12.0	10.9	0.0	7.2	5.3	0.0
	200	24.7	8.1	7.0	0.0	6.2	5.0	0.0
0.9	50	93.4	30.8	26.5	2.0	10.7	6.7	1.6
	100	76.4	22.7	19.9	0.2	8.8	6.7	0.1
	200	47.1	12.6	11.4	0.0	7.3	5.9	0.0

Notes: see Table 1.1

TABLE 3.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 0$, GARCH errors

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
γ	T	$r=0$	1	2	3, 4	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*
0.0	50	81.1	15.8	2.5	0.7	95.0	4.3	0.6	0.1	0.1	95.7	3.6	0.6	0.1	0.1	94.8	4.5	0.6	0.1	0.0	96.7	2.8	0.4	0.1	0.1
	100	89.0	9.5	1.3	0.3	94.1	5.4	0.4	0.1	0.0	94.9	4.7	0.4	0.1	0.0	94.2	5.3	0.4	0.1	0.0	95.6	3.7	0.5	0.2	0.0
	200	91.3	7.7	0.8	0.2	94.0	5.4	0.6	0.1	0.0	95.3	4.2	0.4	0.1	0.0	94.2	5.2	0.5	0.1	0.0	95.4	4.0	0.4	0.2	0.0
0.5	50	60.3	30.4	7.3	2.0	92.3	6.9	0.7	0.2	0.2	93.0	6.0	0.9	0.1	0.1	94.3	5.2	0.5	0.1	0.0	96.0	3.4	0.4	0.2	0.1
	100	80.1	16.7	2.7	0.6	93.0	6.4	0.5	0.2	0.0	93.8	5.6	0.5	0.1	0.0	93.9	5.6	0.5	0.1	0.0	95.3	4.0	0.5	0.2	0.0
	200	88.4	10.1	1.3	0.3	94.0	5.4	0.6	0.0	0.0	94.9	4.6	0.4	0.1	0.0	94.3	5.2	0.5	0.1	0.0	95.4	4.0	0.5	0.2	0.1
0.8	50	20.0	37.4	26.1	16.5	80.5	16.3	2.7	0.5	2.4	82.9	14.3	2.2	0.5	2.3	91.8	7.3	0.8	0.1	0.2	94.8	4.4	0.5	0.3	0.2
	100	52.5	33.4	10.8	3.4	88.1	10.5	1.1	0.3	0.2	89.2	9.6	1.0	0.2	0.2	92.8	6.5	0.6	0.1	0.1	94.7	4.6	0.6	0.1	0.1
	200	75.3	20.2	3.6	1.0	91.9	7.1	0.9	0.1	0.1	93.0	6.1	0.7	0.2	0.0	93.8	5.5	0.6	0.1	0.0	95.0	4.3	0.5	0.3	0.0
0.9	50	6.7	23.4	30.4	39.6	69.3	24.7	4.9	1.1	9.0	73.5	20.8	4.9	0.8	8.0	89.3	9.3	1.1	0.3	2.3	93.3	5.5	0.9	0.4	2.1
	100	23.6	38.3	23.4	14.7	77.4	19.3	2.8	0.6	1.8	80.1	16.8	2.7	0.5	1.8	91.3	7.7	0.9	0.2	0.2	93.3	5.6	0.8	0.3	0.2
	200	52.9	33.5	10.4	3.2	87.4	10.9	1.3	0.3	0.3	88.6	10.0	1.1	0.2	0.3	92.7	6.5	0.7	0.1	0.0	94.2	5.1	0.5	0.3	0.1

Notes: see Table 1.2

TABLE 4.1: EMPIRICAL REJECTION FREQUENCIES OF ASYMPTOTIC AND BOOTSTRAP CO-INTEGRATION RANK TESTS. VAR(2) MODEL WITH RANK $r_0 = 0$, AUTOREGRESSIVE STOCHASTIC VOLATILITY

		$Q_{0,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$	
γ	T	ERF	ERF	ERF	RC	ERF	ERF	RC
0.0	50	33.9	14.5	8.6	0.3	14.4	4.2	0.1
	100	28.5	19.0	8.2	0.0	19.0	5.2	0.0
	200	26.9	21.5	7.3	0.0	21.4	5.6	0.0
0.5	50	51.5	17.3	11.6	1.0	14.4	4.7	0.4
	100	35.7	18.4	8.8	0.1	17.0	4.6	0.1
	200	29.0	19.4	7.4	0.1	19.0	5.8	0.0
0.8	50	82.6	29.8	21.1	3.6	18.4	6.2	2.5
	100	57.2	22.9	12.3	0.9	17.5	5.4	0.9
	200	38.7	19.4	8.6	0.3	17.3	5.6	0.3
0.9	50	93.3	40.2	28.9	6.4	22.4	7.6	6.2
	100	78.9	32.7	19.7	2.4	20.5	6.9	2.4
	200	55.7	23.6	12.0	0.7	18.2	6.4	0.6

Notes: see Table 1.1

TABLE 4.2: SEQUENTIAL PROCEDURES FOR DETERMINING THE CO-INTEGRATION RANK. VAR(2) MODEL WITH RANK $r_0 = 0$, AUTOREGRESSIVE STOCHASTIC VOLATILITY

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
γ	T	$r=0$	1	2	$3,4$	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC
0.0	50	66.1	26.8	5.9	1.3	85.5	12.6	1.6	0.3	0.6	91.4	7.6	0.9	0.2	0.6	85.6	12.5	1.6	0.2	0.5	95.8	3.4	0.5	0.3	0.4
	100	71.5	23.9	3.8	0.8	81.0	16.7	2.0	0.4	0.4	91.8	7.4	0.7	0.1	0.2	81.0	16.6	2.0	0.4	0.4	94.8	4.4	0.7	0.2	0.2
	200	73.1	22.1	4.1	0.7	78.6	18.3	2.6	0.5	0.3	92.7	6.5	0.7	0.0	0.2	78.6	18.2	2.7	0.5	0.3	94.4	4.6	0.8	0.2	0.2
0.5	50	48.6	36.3	11.9	3.2	82.7	15.1	1.9	0.3	1.7	88.4	10.2	1.2	0.2	1.4	85.6	12.7	1.6	0.2	1.1	95.3	3.8	0.6	0.3	0.8
	100	64.3	29.2	5.3	1.2	81.6	15.9	2.1	0.3	0.5	91.2	7.9	0.8	0.1	0.3	83.0	14.9	1.8	0.4	0.5	95.4	3.9	0.5	0.3	0.3
	200	71.0	23.6	4.5	0.9	80.6	16.5	2.5	0.4	0.4	92.6	6.6	0.8	0.1	0.2	81.0	16.3	2.4	0.4	0.4	94.2	4.8	0.7	0.2	0.2
0.8	50	17.4	37.2	28.3	17.1	70.2	24.4	4.5	0.8	6.8	78.9	17.5	3.0	0.6	5.9	81.6	16.1	2.0	0.3	3.4	93.9	5.0	0.8	0.4	3.1
	100	42.8	38.7	14.2	4.3	77.1	19.4	3.2	0.3	1.7	87.7	10.6	1.6	0.2	1.4	82.5	15.2	2.1	0.2	1.3	94.6	4.3	0.7	0.4	1.1
	200	61.3	29.9	7.4	1.4	80.6	16.8	2.3	0.3	0.6	91.4	7.6	0.9	0.1	0.4	82.7	15.1	1.9	0.4	0.6	94.4	4.7	0.7	0.2	0.4
0.9	50	6.7	23.9	30.8	38.7	59.8	31.3	7.4	1.5	15.0	71.1	22.4	5.5	1.0	13.1	77.6	19.3	2.7	0.5	7.8	92.4	6.1	1.0	0.6	7.0
	100	21.1	37.6	26.4	14.9	67.3	26.2	5.6	0.8	5.2	80.3	16.3	2.8	0.6	4.6	79.5	17.5	2.7	0.3	3.2	93.1	5.5	0.9	0.5	2.9
	200	44.3	38.1	13.3	4.4	76.4	19.8	3.4	0.4	1.4	88.0	10.2	1.5	0.3	1.3	81.8	15.9	2.1	0.3	1.1	93.6	5.0	0.9	0.5	1.0

Notes: see Table 1.2

TABLE 5.1: EMPIRICAL REJECTION FREQUENCIES OF ASYMPTOTIC AND BOOTSTRAP CO-INTEGRATION RANK TESTS. VAR(2) MODEL WITH RANK $r_0 = 0$, SINGLE VOLATILITY BREAK

		$Q_{0,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$	
γ	T	ERF	ERF	ERF	RC	ERF	ERF	RC
0.0	50	55.4	28.7	17.9	0.0	28.6	10.1	0.0
	100	46.3	32.2	12.2	0.0	31.8	8.7	0.0
	200	42.7	34.2	8.7	0.0	34.2	7.1	0.0
0.5	50	68.3	28.1	18.1	0.5	24.5	9.9	0.1
	100	54.2	31.3	12.9	0.0	29.1	8.2	0.0
	200	47.2	33.8	9.1	0.0	33.3	6.8	0.0
0.8	50	86.0	33.5	22.1	6.0	22.8	10.6	4.0
	100	69.7	32.1	14.6	0.2	24.8	7.8	0.1
	200	58.1	32.7	9.8	0.0	29.5	6.2	0.0
0.9	50	93.6	41.2	28.3	10.9	24.1	12.3	15.9
	100	83.6	37.8	18.0	2.6	24.1	8.8	2.9
	200	69.6	33.6	11.1	0.2	25.6	6.7	0.1

Notes: see Table 1.1

TABLE 5.2: SEQUENTIAL PROCEDURES FOR DETERMINING THE CO-INTEGRATION RANK. VAR(2) MODEL WITH RANK $r_0 = 0$, SINGLE VOLATILITY BREAK

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
γ	T	$r=0$	1	2	3, 4	$r=0$	1	2	3, 4	RC	$r=0$	1	2	3, 4	RC	$r=0$	1	2	3, 4	RC	$r=0$	1	2	3, 4	RC
0.0	50	44.7	40.9	12.5	2.0	71.3	25.1	3.2	0.4	1.6	82.1	15.3	2.2	0.4	1.2	71.4	25.1	3.1	0.4	1.5	90.0	8.6	1.2	0.3	0.7
	100	53.7	36.3	8.8	1.2	67.8	27.9	4.1	0.3	1.4	87.8	10.7	1.3	0.2	0.6	68.2	27.6	3.9	0.3	1.3	91.3	7.7	0.9	0.2	0.5
	200	57.3	34.2	7.6	0.9	65.8	29.4	4.4	0.4	1.1	91.3	7.7	0.8	0.2	0.3	65.9	29.2	4.5	0.4	1.0	92.9	6.3	0.6	0.2	0.2
0.5	50	31.7	45.8	18.4	4.1	71.9	24.7	3.1	0.2	3.8	81.9	15.8	2.0	0.3	3.0	75.5	21.8	2.5	0.2	2.2	90.1	8.7	1.0	0.2	1.0
	100	45.8	40.2	11.9	2.1	68.7	26.3	4.5	0.4	1.9	87.1	10.8	1.8	0.2	1.0	70.9	24.9	4.0	0.3	1.7	91.8	7.1	1.0	0.1	0.6
	200	52.8	37.1	8.7	1.4	66.3	29.2	4.2	0.4	1.3	90.9	8.0	1.0	0.1	0.4	66.8	28.8	4.1	0.3	1.3	93.2	6.1	0.6	0.1	0.4
0.8	50	14.0	41.7	31.9	12.4	66.5	29.1	4.2	0.2	15.7	77.9	19.0	2.8	0.3	13.5	77.2	20.8	1.9	0.1	8.2	89.4	9.4	1.1	0.2	6.5
	100	30.3	45.8	19.2	4.7	67.9	28.1	3.7	0.3	4.7	85.4	12.8	1.6	0.2	2.6	75.2	22.3	2.4	0.1	2.7	92.2	6.7	0.9	0.1	1.2
	200	41.9	42.5	13.0	2.6	67.3	28.2	4.2	0.4	2.4	90.2	8.7	0.9	0.2	0.9	70.5	25.8	3.4	0.3	1.9	93.8	5.7	0.5	0.1	0.5
0.9	50	6.5	30.2	36.5	26.9	58.8	35.2	5.7	0.3	27.4	71.7	24.1	3.8	0.4	23.7	75.9	21.6	2.3	0.1	21.9	87.7	10.7	1.4	0.2	19.5
	100	16.4	43.5	29.5	10.6	62.3	33.2	4.3	0.3	11.8	82.0	15.8	2.0	0.2	7.8	75.9	21.9	2.0	0.1	6.9	91.2	7.8	0.9	0.1	4.5
	200	30.4	46.2	19.0	4.4	66.4	29.1	4.1	0.4	4.6	88.9	9.7	1.2	0.2	1.9	74.4	23.0	2.5	0.1	2.8	93.3	6.1	0.6	0.1	0.9

Notes: see Table 1.2

TABLE 6.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 1$, IID Gaussian errors

		$Q_{0,T}$	$Q_{1,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$\tilde{Q}^{*\text{iid}}_{1,T}$	$\tilde{Q}^{*\text{w}}_{1,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{1,T}$	$Q^{*\text{w}}_{1,T}$	
δ	T	ERF	ERF	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC
0.0	50	97.6	46.4	59.0	50.5	2.4	9.9	9.4	0.4	56.6	50.8	0.0	4.9	4.5	0.1
	100	100.0	23.6	99.4	99.0	0.1	7.9	7.6	0.0	99.3	99.2	0.0	5.6	4.3	0.0
	200	100.0	13.9	100.0	100.0	0.0	6.1	5.9	0.0	100.0	100.0	0.0	5.4	4.8	0.0
0.1	50	97.6	46.2	56.3	47.6	7.2	10.0	9.8	0.4	55.6	49.3	0.0	4.7	4.4	0.1
	100	99.9	23.4	99.1	98.7	2.4	8.0	7.7	0.0	99.3	99.1	0.0	5.5	4.2	0.0
	200	100.0	13.9	100.0	100.0	0.5	6.1	5.9	0.0	100.0	100.0	0.0	5.4	4.8	0.0
0.2	50	97.2	46.1	48.5	40.7	24.1	9.9	9.2	0.7	52.7	46.2	0.0	4.7	4.3	0.1
	100	99.9	23.1	98.2	97.2	29.0	8.6	8.2	0.0	99.1	98.8	0.0	5.6	4.2	0.0
	200	100.0	13.6	100.0	100.0	34.4	6.8	6.6	0.0	100.0	100.0	0.0	5.5	4.8	0.0
0.3	50	97.1	46.0	41.7	35.3	48.8	10.4	10.2	1.6	50.2	44.0	0.0	4.7	4.2	0.1
	100	100.0	22.7	94.6	92.7	73.4	9.2	9.1	0.0	98.9	98.5	0.0	5.4	4.3	0.0
	200	100.0	13.2	100.0	100.0	91.6	9.1	8.5	0.0	100.0	100.0	0.0	5.5	4.7	0.0

Notes: see Table 1.1

TABLE 6.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 1$, IID Gaussian errors

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
δ	T	$r=0$	1	2	$3,4$	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC
0.0	50	2.8	51.2	32.9	13.1	41.0	49.2	8.5	1.3	3.8	49.5	41.4	8.0	1.2	3.5	43.4	51.7	4.3	0.6	0.3	49.2	46.4	3.6	0.8	0.3
	100	0.1	78.0	18.1	3.8	0.7	91.4	7.2	0.7	0.3	1.0	91.4	6.8	0.8	0.3	0.7	93.7	5.1	0.5	0.1	0.8	94.9	3.5	0.8	0.1
	200	0.0	86.8	11.6	1.6	0.0	93.9	5.4	0.6	0.1	0.0	94.1	5.3	0.6	0.0	0.0	94.6	4.9	0.5	0.0	0.0	95.2	4.1	0.6	0.0
0.1	50	2.6	51.0	32.6	13.8	43.7	46.4	8.4	1.5	8.5	52.4	38.1	8.3	1.3	8.3	44.4	50.9	4.1	0.5	0.2	50.7	45.0	3.4	0.9	0.2
	100	0.1	76.8	19.1	4.0	0.9	91.1	7.4	0.7	2.6	1.3	91.0	6.9	0.8	2.7	0.8	93.8	5.1	0.4	0.1	0.9	94.9	3.4	0.9	0.1
	200	0.0	86.5	11.8	1.7	0.0	93.9	5.5	0.6	0.5	0.0	94.2	5.3	0.6	0.6	0.0	94.7	4.8	0.5	0.0	0.0	95.2	4.2	0.6	0.0
0.2	50	2.4	51.0	32.4	14.2	51.5	38.7	8.4	1.5	25.3	59.3	31.8	7.6	1.3	25.2	47.3	48.0	4.1	0.6	0.2	53.8	42.1	3.3	0.8	0.2
	100	0.0	76.4	19.5	4.0	1.8	89.6	7.8	0.8	29.3	2.8	88.9	7.4	0.9	29.1	0.9	93.6	5.1	0.4	0.1	1.2	94.6	3.4	0.8	0.1
	200	0.0	86.1	12.1	1.8	0.0	93.2	6.1	0.7	34.3	0.0	93.4	5.8	0.8	34.4	0.0	94.5	5.0	0.5	0.0	0.0	95.2	4.3	0.6	0.0
0.3	50	2.4	51.2	32.5	13.9	58.3	31.6	8.4	1.7	49.7	64.7	25.6	8.0	1.7	49.4	49.8	45.6	4.1	0.6	0.2	56.0	40.0	3.2	0.9	0.2
	100	0.0	76.3	19.5	4.1	5.5	85.3	8.4	0.9	73.6	7.3	83.6	8.1	1.0	73.4	1.1	93.5	5.0	0.4	0.1	1.5	94.2	3.6	0.8	0.1
	200	0.0	86.1	12.1	1.8	0.0	90.9	8.0	1.1	91.5	0.0	91.6	7.3	1.1	91.6	0.0	94.5	5.0	0.5	0.0	0.0	95.3	4.2	0.6	0.0

Notes: see Table 1.2

TABLE 7.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 1$, IID $t(5)$ errors

		$Q_{0,T}$	$Q_{1,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$\tilde{Q}^{*\text{iid}}_{1,T}$	$\tilde{Q}^{*\text{w}}_{1,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{1,T}$	$Q^{*\text{w}}_{1,T}$	
δ	T	ERF	ERF	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC
0.0	50	97.8	46.5	56.1	49.0	2.3	10.4	9.7	0.4	54.2	49.5	0.0	5.0	4.8	0.1
	100	100.0	24.5	99.2	98.6	0.1	7.8	7.9	0.0	99.3	98.6	0.0	5.5	4.9	0.0
	200	100.0	14.0	100.0	100.0	0.0	6.0	6.5	0.0	100.0	100.0	0.0	5.0	5.1	0.0
0.1	50	97.6	46.4	54.6	46.5	6.6	10.2	9.6	0.4	53.4	48.7	0.0	4.8	4.4	0.2
	100	100.0	24.1	99.1	98.2	2.4	7.7	7.9	0.0	99.2	98.7	0.0	5.2	4.8	0.0
	200	100.0	14.0	100.0	100.0	0.4	5.9	6.5	0.0	100.0	100.0	0.0	5.2	5.2	0.0
0.2	50	97.3	46.5	48.1	40.4	24.3	9.6	9.3	0.8	51.3	46.4	0.1	4.5	4.4	0.2
	100	100.0	23.9	97.9	96.6	30.1	8.9	8.7	0.0	99.0	98.3	0.0	5.1	4.6	0.0
	200	100.0	13.6	100.0	100.0	34.3	6.6	6.9	0.0	100.0	100.0	0.0	5.1	5.2	0.0
0.3	50	96.9	46.6	42.2	35.6	49.5	9.6	9.2	1.8	48.8	44.8	0.0	4.5	4.3	0.2
	100	100.0	23.2	93.8	91.8	73.6	10.1	9.3	0.0	98.9	98.2	0.0	5.2	4.6	0.0
	200	100.0	13.2	100.0	100.0	91.8	8.4	9.1	0.0	100.0	100.0	0.0	5.2	5.2	0.0

Notes: see Table 1.1

TABLE 7.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 1$, IID $t(5)$ errors

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
δ	T	$r=0$	1	2	3, 4	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*
0.0	50	2.9	51.5	32.6	13.1	43.9	45.8	9.0	1.3	3.7	51.0	39.7	7.9	1.5	3.5	45.8	49.2	4.4	0.6	0.3	50.5	45.2	3.7	0.7	0.3
	100	0.0	76.8	19.0	4.1	0.8	91.4	7.1	0.8	0.3	1.4	90.7	7.0	0.9	0.4	0.7	93.8	4.9	0.5	0.1	1.4	93.8	4.0	0.9	0.2
	200	0.0	87.1	11.0	1.9	0.0	94.0	5.4	0.6	0.0	0.0	93.5	5.9	0.6	0.1	0.0	95.0	4.6	0.5	0.0	0.0	94.9	4.4	0.7	0.1
0.1	50	2.4	51.5	32.8	13.4	45.4	44.5	8.9	1.2	7.7	53.5	37.3	7.8	1.4	8.0	46.6	48.7	4.3	0.5	0.2	51.3	44.6	3.5	0.7	0.3
	100	0.0	76.1	19.7	4.2	0.9	91.4	7.0	0.7	2.5	1.8	90.3	7.0	1.0	2.7	0.8	93.9	4.7	0.5	0.1	1.3	93.9	3.9	0.9	0.2
	200	0.0	86.6	11.5	1.9	0.0	94.1	5.3	0.6	0.4	0.0	93.5	5.9	0.6	0.4	0.0	94.8	4.8	0.5	0.0	0.0	94.8	4.4	0.8	0.1
0.2	50	2.2	51.5	32.6	13.7	51.9	38.6	8.3	1.2	25.2	59.6	31.7	7.2	1.5	25.2	48.7	46.9	3.9	0.5	0.3	53.6	42.4	3.3	0.7	0.3
	100	0.0	75.7	19.7	4.6	2.1	88.9	8.0	0.9	29.9	3.4	88.0	7.6	1.1	30.7	1.0	93.9	4.6	0.5	0.1	1.7	93.7	3.8	0.9	0.2
	200	0.0	86.0	12.0	2.0	0.0	93.4	5.8	0.7	34.4	0.0	93.1	6.2	0.8	34.2	0.0	94.9	4.7	0.5	0.0	0.0	94.8	4.4	0.8	0.0
0.3	50	2.2	51.3	32.6	14.0	57.8	32.9	7.7	1.5	50.3	64.4	26.8	7.2	1.6	50.4	51.2	44.3	4.0	0.4	0.3	55.2	40.9	3.2	0.7	0.2
	100	0.0	75.5	19.9	4.5	6.2	83.7	8.9	1.3	73.6	8.3	82.5	8.0	1.2	73.7	1.1	93.6	4.8	0.5	0.1	1.8	93.6	3.7	0.9	0.1
	200	0.0	86.0	12.0	2.0	0.0	91.6	7.7	0.7	91.8	0.0	90.9	8.2	1.0	91.9	0.0	94.8	4.7	0.6	0.0	0.0	94.8	4.4	0.8	0.0

Notes: see Table 1.2

TABLE 8.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 1$, GARCH errors

		$Q_{0,T}$	$Q_{1,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$\tilde{Q}^{*\text{iid}}_{1,T}$	$\tilde{Q}^{*\text{w}}_{1,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{1,T}$	$Q^{*\text{w}}_{1,T}$	
δ	T	ERF	ERF	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC
0.0	50	97.3	47.7	58.2	50.4	2.6	10.5	9.2	0.4	55.6	51.1	0.1	5.4	4.0	0.2
	100	100.0	24.8	99.1	98.6	0.1	8.6	7.7	0.0	99.2	98.8	0.0	6.0	5.1	0.0
	200	100.0	14.3	100.0	100.0	0.0	7.2	6.1	0.0	100.0	100.0	0.0	6.2	4.9	0.0
0.1	50	97.5	47.3	55.3	47.1	7.4	10.4	9.3	0.5	54.7	49.6	0.0	5.0	4.0	0.1
	100	100.0	24.9	98.8	98.2	2.8	8.8	7.7	0.0	99.1	98.7	0.0	6.2	5.1	0.0
	200	100.0	14.3	100.0	100.0	0.5	7.0	6.0	0.0	100.0	100.0	0.0	6.1	4.9	0.0
0.2	50	97.2	46.8	49.6	41.7	25.1	10.3	9.2	0.7	52.8	47.3	0.0	4.8	4.1	0.2
	100	100.0	24.1	97.5	96.3	30.6	9.3	8.5	0.0	98.6	98.2	0.0	6.3	5.1	0.0
	200	100.0	14.1	100.0	100.0	34.5	7.6	6.8	0.0	100.0	100.0	0.0	6.1	4.7	0.0
0.3	50	96.7	46.7	44.3	37.9	48.8	9.6	9.0	1.7	50.8	45.5	0.0	4.8	3.9	0.1
	100	99.9	23.5	91.6	88.8	72.8	10.5	9.5	0.0	98.0	97.4	0.0	6.3	5.1	0.0
	200	100.0	13.8	100.0	100.0	91.0	9.8	8.1	0.0	100.0	100.0	0.0	6.0	4.7	0.0

Notes: see Table 1.1

TABLE 8.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 1$, GARCH errors

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
δ	T	$r=0$	1	2	3, 4	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*	$r=0$	1	2	3, 4	*RC*
0.0	50	3.5	49.7	34.0	12.8	41.9	47.7	9.0	1.5	3.6	49.6	41.4	7.7	1.3	4.1	44.4	50.2	4.7	0.6	0.2	48.9	47.2	3.1	0.8	0.2
	100	0.1	76.1	19.8	3.9	0.9	90.6	7.7	0.9	0.3	1.4	90.9	6.9	0.8	0.3	0.8	93.2	5.5	0.6	0.0	1.2	93.8	4.2	0.9	0.1
	200	0.0	86.3	11.7	2.0	0.0	92.8	6.5	0.7	0.2	0.0	93.9	5.5	0.6	0.1	0.0	93.8	5.6	0.6	0.1	0.0	95.1	4.0	0.9	0.1
0.1	50	2.9	50.4	33.8	12.9	44.7	45.1	8.9	1.3	8.4	52.9	38.2	7.5	1.5	8.9	45.4	49.7	4.4	0.5	0.2	50.4	45.7	3.2	0.7	0.3
	100	0.0	75.6	20.3	4.1	1.2	90.0	7.9	0.9	2.9	1.9	90.5	6.8	0.9	3.0	1.0	92.9	5.7	0.5	0.1	1.3	93.7	4.2	0.9	0.1
	200	0.0	85.9	12.0	2.1	0.0	93.0	6.3	0.7	0.7	0.0	94.0	5.4	0.6	0.6	0.0	93.9	5.6	0.6	0.1	0.0	95.1	4.0	0.9	0.0
0.2	50	2.7	50.0	34.1	13.3	50.4	39.5	8.9	1.2	26.2	58.3	32.8	7.6	1.3	25.8	47.3	48.0	4.4	0.4	0.2	52.7	43.4	3.1	0.8	0.2
	100	0.0	75.2	20.5	4.3	2.6	88.1	8.2	1.1	30.9	3.7	87.9	7.4	1.1	30.5	1.4	92.3	5.8	0.5	0.1	1.8	93.2	4.2	0.9	0.0
	200	0.0	85.7	12.1	2.2	0.0	92.4	6.8	0.8	34.7	0.0	93.2	6.1	0.7	34.5	0.0	93.9	5.5	0.6	0.1	0.0	95.3	4.0	0.8	0.1
0.3	50	2.7	49.6	34.3	13.5	55.7	34.9	8.0	1.4	49.6	62.1	29.3	7.4	1.2	49.3	49.2	46.1	4.2	0.5	0.2	54.5	41.7	3.1	0.7	0.2
	100	0.0	75.2	20.2	4.6	8.4	81.0	9.3	1.2	72.9	11.2	79.3	8.4	1.0	72.8	2.0	91.7	5.8	0.6	0.1	2.6	92.3	4.2	1.0	0.0
	200	0.0	85.7	12.1	2.2	0.0	90.2	8.9	0.9	91.1	0.0	91.9	7.5	0.7	91.0	0.0	94.0	5.5	0.5	0.1	0.0	95.3	3.9	0.8	0.1

Notes: see Table 1.2

TABLE 9.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 1$, Autoregressive Stochastic Volatility

		$Q_{0,T}$	$Q_{1,T}$	$\tilde{Q}^{*\mathrm{iid}}_{0,T}$	$\tilde{Q}^{*\mathrm{w}}_{0,T}$		$\tilde{Q}^{*\mathrm{iid}}_{1,T}$	$\tilde{Q}^{*\mathrm{w}}_{1,T}$		$Q^{*\mathrm{iid}}_{0,T}$	$Q^{*\mathrm{w}}_{0,T}$		$Q^{*\mathrm{iid}}_{1,T}$	$Q^{*\mathrm{w}}_{1,T}$	
δ	T	ERF	ERF	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC
0.0	50	96.2	52.4	60.1	46.8	7.1	16.0	12.0	4.2	58.6	43.8	1.8	10.5	6.6	3.4
	100	99.6	36.2	96.8	90.1	1.5	18.1	9.9	1.1	96.7	90.6	0.7	15.5	6.4	1.1
	200	100.0	26.0	100.0	99.5	0.3	16.2	7.2	0.3	100.0	99.4	0.2	15.1	5.4	0.3
0.1	50	95.9	51.6	59.7	46.0	15.2	15.3	11.3	4.9	59.5	45.0	1.7	9.9	6.8	3.3
	100	99.5	35.5	95.8	88.8	8.5	18.0	9.9	0.9	96.5	90.1	0.6	15.0	6.2	1.0
	200	100.0	25.9	100.0	99.5	4.1	16.1	7.2	0.3	100.0	99.4	0.2	14.9	5.2	0.2
0.2	50	94.3	49.5	58.3	46.9	31.3	14.9	11.0	7.6	58.8	45.9	1.6	9.1	6.4	3.0
	100	98.8	34.5	91.1	82.1	33.6	18.3	10.1	1.2	94.2	86.3	0.6	14.3	6.2	0.9
	200	99.9	25.5	99.8	98.4	37.0	17.0	7.4	0.2	99.8	98.9	0.2	14.8	5.3	0.2
0.3	50	92.7	48.3	57.8	48.3	49.3	14.3	10.0	10.9	57.7	46.5	1.8	8.5	5.9	3.2
	100	97.7	33.2	83.3	75.2	65.8	18.3	10.1	2.1	90.9	82.5	0.6	14.0	5.8	0.9
	200	99.8	25.1	98.4	94.6	80.2	19.0	8.4	0.2	99.7	98.0	0.2	14.6	5.4	0.2

Notes: see Table 1.1

TABLE 9.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 1$, Autoregressive Stochastic Volatility

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
δ	T	$r=0$	1	2	$3,4$	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC
0.0	50	7.0	43.9	34.2	14.9	39.9	44.1	13.9	2.0	9.6	53.2	35.3	10.1	1.4	8.9	41.4	48.2	9.3	1.1	3.9	56.2	38.9	3.8	1.1	3.4
	100	2.2	64.6	27.0	6.2	3.2	78.7	15.9	2.2	2.5	9.9	80.3	8.8	1.0	2.2	3.3	81.2	13.8	1.7	1.7	9.4	84.6	5.0	1.1	1.4
	200	0.1	75.0	21.2	3.7	0.0	83.8	14.3	1.9	0.7	0.5	92.3	6.4	0.8	0.5	0.0	84.9	13.5	1.7	0.7	0.6	94.1	4.5	0.9	0.5
0.1	50	5.7	43.5	35.3	15.4	40.3	44.6	13.1	2.1	17.4	54.0	35.2	9.5	1.3	17.2	40.5	49.7	8.8	1.0	3.7	55.0	40.1	3.9	1.1	3.0
	100	1.2	64.2	27.9	6.8	4.3	77.7	15.9	2.2	9.5	11.2	79.0	8.8	1.0	9.0	3.5	81.5	13.2	1.8	1.6	9.9	84.2	4.8	1.0	1.4
	200	0.0	74.5	21.7	3.9	0.0	83.9	14.3	1.8	4.5	0.5	92.3	6.3	0.9	4.3	0.1	85.1	13.1	1.8	0.6	0.6	94.2	4.4	0.8	0.4
0.2	50	4.4	43.5	36.2	15.9	41.7	43.6	12.7	1.9	33.4	53.1	36.3	9.2	1.4	33.3	41.2	49.7	8.0	1.0	3.4	54.1	41.2	3.6	1.1	2.7
	100	0.5	63.6	28.9	7.0	8.9	72.9	15.9	2.3	34.1	17.9	72.1	8.9	1.1	34.1	5.8	79.9	12.6	1.7	1.4	13.7	80.6	4.8	1.0	1.1
	200	0.0	74.2	21.7	4.1	0.2	82.8	14.9	2.1	37.1	1.6	91.0	6.5	1.0	37.1	0.2	85.0	13.1	1.8	0.5	1.1	93.7	4.4	0.9	0.4
0.3	50	3.8	43.7	36.4	16.1	42.2	43.6	12.3	1.9	51.3	51.7	38.7	8.2	1.5	50.8	42.3	49.2	7.6	0.8	3.5	53.5	42.2	3.3	1.0	3.0
	100	0.4	63.4	29.0	7.2	16.7	65.1	15.9	2.4	66.4	24.8	65.3	8.8	1.1	66.2	9.1	76.9	12.3	1.7	1.4	17.6	77.2	4.3	1.0	1.1
	200	0.0	74.0	21.7	4.3	1.7	79.4	16.7	2.3	80.3	5.4	86.2	7.4	1.0	80.3	0.3	85.1	13.0	1.6	0.6	2.0	92.6	4.5	0.9	0.4

Notes: see Table 1.2

TABLE 10.1: Empirical Rejection Frequencies of Asymptotic and Bootstrap Co-integration Rank Tests. VAR(2) Model with rank $r_0 = 1$, Single Volatility Break

		$Q_{0,T}$	$Q_{1,T}$	$\tilde{Q}^{*\text{iid}}_{0,T}$	$\tilde{Q}^{*\text{w}}_{0,T}$		$\tilde{Q}^{*\text{iid}}_{1,T}$	$\tilde{Q}^{*\text{w}}_{1,T}$		$Q^{*\text{iid}}_{0,T}$	$Q^{*\text{w}}_{0,T}$		$Q^{*\text{iid}}_{1,T}$	$Q^{*\text{w}}_{1,T}$	
δ	T	ERF	ERF	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC	ERF	ERF	RC
0.0	50	97.5	53.6	65.1	53.2	14.8	12.8	9.9	8.0	65.6	49.8	2.5	8.9	6.2	7.1
	100	99.9	41.8	98.9	95.5	2.5	18.8	10.0	0.3	98.9	95.1	0.1	15.4	6.8	0.2
	200	100.0	34.2	100.0	100.0	0.1	19.9	7.5	0.0	100.0	100.0	0.0	18.7	5.9	0.0
0.1	50	97.3	53.6	63.3	52.0	19.8	12.5	9.8	8.3	65.9	49.0	2.6	8.8	6.1	6.6
	100	99.9	41.9	98.8	94.6	9.1	18.7	10.3	0.2	98.9	94.5	0.1	15.6	6.9	0.2
	200	100.0	34.3	100.0	100.0	3.4	20.0	7.6	0.0	100.0	100.0	0.0	18.7	5.9	0.0
0.2	50	97.2	52.3	60.1	47.9	31.4	11.7	8.9	9.2	64.8	47.3	2.6	8.6	5.7	6.4
	100	99.9	41.4	97.8	91.9	31.5	18.7	10.2	0.3	98.6	94.0	0.1	15.5	6.7	0.1
	200	100.0	34.3	100.0	100.0	34.5	20.9	7.7	0.0	100.0	100.0	0.0	18.9	5.6	0.0
0.3	50	96.9	51.4	57.2	45.5	45.7	11.3	8.4	11.3	63.4	46.5	2.7	8.1	5.4	6.4
	100	99.9	41.1	95.9	86.6	61.2	18.7	10.3	0.5	98.7	93.4	0.1	15.8	6.6	0.2
	200	100.0	34.0	100.0	100.0	79.9	21.5	8.6	0.0	100.0	100.0	0.0	18.4	5.7	0.0

Notes: see Table 1.1

TABLE 10.2: Sequential Procedures for Determining the Co-integration Rank. VAR(2) Model with rank $r_0 = 1$, Single Volatility Break

		ASYMPTOTIC				Unrestricted, IID BS					Unrestricted, Wild BS					Restricted, IID BS [CRT]					Restricted, Wild BS [CRT]				
δ	T	$r=0$	1	2	$3,4$	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC	$r=0$	1	2	$3,4$	RC
0.0	50	2.5	43.9	40.5	13.1	34.9	52.4	11.9	0.9	22.4	46.9	43.5	8.8	0.9	21.5	34.4	56.7	8.4	0.5	8.9	50.2	43.7	5.5	0.6	7.9
	100	0.1	58.1	35.1	6.8	1.1	80.1	17.7	1.0	6.6	4.5	85.4	9.3	0.8	5.1	1.1	83.5	14.7	0.8	2.8	4.9	88.3	6.3	0.5	1.4
	200	0.0	65.8	29.4	4.8	0.0	80.1	18.2	1.7	2.5	0.0	92.5	6.8	0.7	1.2	0.0	81.3	17.1	1.5	2.0	0.0	94.1	5.3	0.6	0.7
0.1	50	2.7	43.8	40.7	12.9	36.7	50.9	11.7	0.8	26.7	48.0	42.6	8.6	0.9	25.8	34.1	57.1	8.3	0.5	8.3	51.0	43.1	5.3	0.6	7.3
	100	0.1	58.0	35.0	6.9	1.2	80.2	17.6	1.0	13.0	5.4	84.4	9.5	0.7	11.3	1.1	83.3	14.7	0.9	2.9	5.5	87.7	6.4	0.5	1.5
	200	0.0	65.7	29.6	4.7	0.0	80.0	18.4	1.7	5.7	0.0	92.5	6.8	0.7	4.4	0.0	81.3	17.2	1.5	2.0	0.0	94.2	5.3	0.6	0.7
0.2	50	2.8	44.9	39.8	12.5	39.9	48.5	10.9	0.7	37.4	52.1	39.4	7.7	0.8	36.6	35.2	56.2	8.1	0.5	8.2	52.7	41.7	5.2	0.4	7.2
	100	0.1	58.5	34.6	6.8	2.2	79.1	17.5	1.2	34.5	8.1	81.7	9.3	0.9	33.3	1.4	83.2	14.6	0.9	2.9	6.0	87.3	6.2	0.5	1.6
	200	0.0	65.8	29.6	4.7	0.0	79.1	19.1	1.7	36.0	0.0	92.3	6.8	0.9	35.3	0.0	81.1	17.5	1.4	1.9	0.0	94.4	5.1	0.5	0.7
0.3	50	3.1	45.5	39.3	12.1	42.8	46.1	10.4	0.7	50.5	54.5	37.3	7.4	0.9	49.9	36.6	55.3	7.7	0.4	8.7	53.5	41.2	5.0	0.3	7.3
	100	0.1	58.8	34.5	6.6	4.1	77.1	17.4	1.3	62.9	13.5	76.3	9.4	0.9	62.2	1.3	83.0	14.8	0.9	3.0	6.6	86.8	6.0	0.6	1.7
	200	0.0	66.0	29.4	4.6	0.0	78.5	19.9	1.7	80.4	0.0	91.4	7.7	0.9	80.1	0.0	81.6	17.1	1.3	2.0	0.0	94.3	5.1	0.6	0.7

Notes: see Table 1.2